ISRAEL JOURNAL OF MATHEMATICS **162** (2007), 109–149
DOI: 10.1007/s11856-007-0091-x

CONVERGENCE THEOREMS AND TAUBERIAN THEOREMS FOR FUNCTIONS AND SEQUENCES IN BANACH SPACES AND BANACH LATTICES

BY

Yuan-Chuan Li*

Department of Applied Mathematics, National Chung-Hsing University
Taichung, 402 Taiwan
e-mail: ycli@fish.amath.nchu.edu.tw

AND

Ryotaro Sato**

Department of Mathematics, Okayama University
Okayama, 700-8530 Japan
e-mail: satoryot@math.okayama-u.ac.jp

AND

Sen-Yen Shaw*

Graduate School of Engineering, Lunghwa University of Science and Technology
Gueishan, Taoyuan, 333 Taiwan
e-mail: shaw@math.ncu.edu.tw

ABSTRACT

We prove generalized convergence theorems and Tauberian theorems for vector-valued functions and sequences of growth order $\gamma - 1$ with $\gamma > 0$ and for positive functions and sequences in Banach lattices. Then the general results are applied to obtain some interesting particular Tauberian results for various examples of operator semigroups. Among them are mean ergodic theorems for Cesàro-mean-bounded semigroups (discrete and continuous) of operators and for semigroups of positive operators.

* Research supported in part by the National Science Council of Taiwan.
** Current address: 19-18, Higashi-hongo 2-chome, Midori-ku, 226-0002 Japan.
Received April 18, 2004 and in revised form June 2, 2006

1. Introduction

Let X be a Banach space, and $x : [0, \infty) \to X$ be a locally integrable function. It is well-known that the existence of the Cesàro limit $y := \lim_{t\to\infty} t^{-1} \int_0^t x(s)ds$ implies that the Abel limit $\lim_{\lambda\downarrow 0} \lambda \int_0^\infty e^{-\lambda t} x(t)dt$ also exists and equals y. Similarly, if for a sequence $\{x_n\}_{n=0}^\infty \subset X$ the Cesàro limit $y := \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} x_k$ exists, then the Abel limit $\lim_{r\uparrow 1}(1-r)\sum_{n=0}^\infty r^n x_n = y$. In general, the existence of the Abel limit does not guarantee the existence of the Cesàro limit. For example, it is shown in [6, p. 8] that if $x_n := 4(-1)^n[n/2] - 1$ for $n \geq 1$, where $[n/2]$ denotes the largest integer not exceeding $n/2$, then $n^{-1}\sum_{k=1}^n x_k = (-1)^n$, but $\lim_{r\uparrow 1}(1-r)\sum_{n=1}^\infty r^n x_n$ exists.

The Tauberian theorem of Hardy and Littlewood is a useful tool in summability theory and ergodic theory [7, Chap. 18]. It states that if $x(\cdot)$ (resp. $\{x_n\}_{n=0}^\infty$) is bounded, or is positive in a Banach lattice, then the existence of the Abel limit also implies the existence of the Cesàro limit, and the two limits coincide (cf. [7], [6, Theorem 3.3]). The convergence rate of Cesàro limit and Abel limit has been an interesting subject. See e.g. [1] and [17]. Recently, in [11], rates of growth of $\|\sum_{k=1}^n P^k f\|_2$ for P a Markov operator are used for a central limit theorem.

The purpose of this paper is to generalize the above convergence theorem

$$\lim_{t\to\infty} t^{-\gamma}\int_0^t x(s)ds \quad\text{and}\quad \lim_{\lambda\downarrow 0}\frac{\lambda^\gamma}{\Gamma(\gamma+1)}\int_0^\infty e^{-\lambda t}x(t)dt$$

$$\Big(\text{resp. } \lim_{n\to\infty} n^{-\gamma}\sum_{k=0}^{n-1} x_k \quad\text{and}\quad \lim_{r\uparrow 1}\frac{(1-r)^\gamma}{\Gamma(\gamma+1)}\sum_{n=0}^\infty r^n x_n\Big).$$

Section 2 is concerned with convergence theorems for the case $\gamma > -1$, Section 3 treats Tauberian theorems for functions (resp. sequences) for which $t^{-\gamma}\int_0^t x(s)ds$ (resp. $n^{-\gamma}\sum_{k=0}^{n-1} x_k$) is bounded and feebly oscillating, and Section 4 proves Tauberian theorems for positive functions and positive sequences in Banach lattices for the case $\gamma \geq 0$. The main results in these three sections can be summarized as follows:

For $\gamma > -1$, if $y := \lim_{t\to\infty} t^{-\gamma}\int_0^t x(s)ds$ (resp. $:= \lim_{n\to\infty} n^{-\gamma}\sum_{k=0}^{n-1} x_k$) exists, then $\lim_{\lambda\downarrow 0}\frac{\lambda^\gamma}{\Gamma(\gamma+1)}\int_0^\infty e^{-\lambda t}x(t)dt$ (resp. $\lim_{r\uparrow 1}\frac{(1-r)^\gamma}{\Gamma(\gamma+1)}\sum_{n=0}^\infty r^n x_n$) $= y$. When $t^{-\gamma}\int_0^t x(s)ds$ (resp. $\{n^{-\gamma}\sum_{k=0}^{n-1} x_k\}$) is bounded and feebly oscillating, or when $\gamma > 0$ and $\|x(t)\| = O(t^{\gamma-1})(t\to\infty)$ (resp. $\|x_n\| = O(n^{\gamma-1})$), or when $\gamma \geq 0$ and $x(\cdot)$ (resp. $\{x_n\}_{n=0}^\infty$) is positive in a Banach lattice, the converse implication is also true.

Applications of the general results in Sections 2 and 3 to discrete semigroup $\{T^n\}$ and continuous semigroups $\{T(t); t \geq 0\}$ of operators will be given in Section 5. We obtain convergence theorem (Proposition 5.1) and Tauberian theorem (Propositions 5.2) between

$$\lim_{n\to\infty} n^{-\alpha-1}\sum_{k=0}^{n-1} T^k x\left(\text{resp. } \lim_{t\to\infty} t^{-\alpha-1}\int_0^t T(s)xds\right)$$

and

$$\lim_{r\uparrow 1}\frac{(1-r)^{\alpha+1}}{\Gamma(\alpha+2)}\sum_{n=0}^{\infty} r^n T^n x\left(\text{resp. } \lim_{\lambda\downarrow 0}\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+2)}\int_0^\infty e^{-\lambda t}T(t)xdt\right)$$

for $\alpha > -2$. Propositions 5.3 and 5.4 present particular properties for the cases $-1 < \alpha < 0$ and $\alpha = 0$, respectively. Section 6 will consist of applications of the general results in Section 4 to semigroups of positive operators. Proposition 6.1 is a Tauberian theorem dealing with the above limits for $\alpha > -1$. Proposition 6.2, a mean ergodic theorem for positive semigroup, is a specialization of Proposition 6.1 for the case $\alpha = 0$. We have exhibited nine illustrating examples scattered in Sections 2, 3, 5, and 6.

A convergence theorem is a special case of a ratio limit theorem (cf. [13]). Naturally, we are interested in generalizing the convergence theorems and Tauberian theorems in Sections 2 and 4 to ratio limit theorems and ratio Tauberian theorems for functions in Banach lattices. Results in this respect will appear in [21].

2. Cesàro mean convergence implies Abel mean convergence

In this section we deduce Abel mean convergence from Cesàro mean convergence. We first prove the following lemmas.

Lemma 2.1: *Suppose $h \in L^1((0,\infty))$ is piecewise continuous on $(0,\infty)$ and has the property that there are two numbers $b > a > 0$ such that h is monotonic on $(0,a)$ and (b,∞). Then*

(i) $\sum_{n=1}^{\infty} h(\lambda n)$ *converges absolutely for all* $\lambda > 0$;

(ii) $\lim_{\lambda\downarrow 0}\lambda\sum_{n=1}^{\infty}|h(\lambda n) - \int_n^{n+1} h(\lambda t)dt| = 0$;

(iii) *if* $f\colon [0,\infty) \to X$ *is a bounded step function satisfying* $f(t) = f([t])$ *for all* $t \geq 0$, *then*

$$\lim_{\lambda\downarrow 0}\left\|\lambda\sum_{n=1}^{\infty} h(\lambda n)f(n) - \lambda\int_0^{\infty} h(\lambda t)f(t)dt\right\| = 0;$$

(iv) *under the additional assumption that* h *is positive, if* $\{u_n\}_{n=1}^{\infty} \subset X$ *converges to* $u \in X$, *then*

$$\lim_{\lambda\downarrow 0}\left\|\lambda\sum_{n=1}^{\infty} h(\lambda n)u_n - \lambda\int_0^{\infty} h(\lambda t)dt\cdot u\right\| = 0.$$

Proof: Since $|h|$ has to be non-increasing on $[b,\infty)$, (i) follows from the integral test.

(ii) Since h is monotonic on $(0,a)$, there exists $a' \in (0,a)$ such that $|h|$ is monotonic on $(0,a')$. Thus, for $0 < \lambda < a'/2$ we have

$$\lambda\min\{|h(\lambda)|,|h(2\lambda)|\} \leq \int_{\lambda}^{2\lambda} |h(t)|dt \to 0 \quad \text{as } \lambda\downarrow 0$$

because $h \in L^1((0,\infty))$. Hence $\lambda h(\lambda) \to 0$ as $\lambda \downarrow 0$. This fact will be used later.

Let $b' > b$, and let $\lambda > 0$ be so small that $a' < ([a'/\lambda]+1)\lambda < a$ and $b < \lambda[b'/\lambda] \leq b'$. Since $([a'/\lambda]+1)\lambda \to a'$ and $([b'/\lambda]-1)\lambda \to b'$ as $\lambda \downarrow 0$, we have

$$\begin{aligned}
&\lambda\sum_{n=[a'/\lambda]+1}^{[b'/\lambda]-1} |h(\lambda n) - \int_n^{n+1} h(\lambda t)dt| \\
&\quad\leq \lambda\sum_{n=[a'/\lambda]+1}^{[b'/\lambda]-1}\Big(\sup_{n\leq t_n\leq n+1} h(\lambda t_n) - \inf_{n\leq s_n\leq n+1} h(\lambda s_n)\Big) \\
&\qquad + (([a'/\lambda]+1)\lambda - a')\Big(\sup_{a'\leq t\leq([a'/\lambda]+1)\lambda} h(t) - \inf_{a'\leq s\leq([a'/\lambda]+1)\lambda} h(s)\Big) \\
&\qquad + (b' - ([b'/\lambda]-1)\lambda)\Big(\sup_{[b'/\lambda]\lambda\leq t\leq b'} h(t) - \inf_{[b'/\lambda]\lambda\leq t\leq b'} h(s)\Big) \\
&\quad= U(P_\lambda,h) - L(P_\lambda,h) \to \int_{a'}^{b'} h(t)dt - \int_{a'}^{b'} h(t)dt = 0 \quad \text{as } \lambda\downarrow 0,
\end{aligned}$$

where P_λ is the partition $\{a', ([a'/\lambda]+1)\lambda, ([a'/\lambda]+2)\lambda, \ldots, ([b'/\lambda]-1)\lambda, b'\}$ of $[a',b']$ and $U(P_\lambda,h)$ and $L(P_\lambda,h)$ are the upper and lower Riemann sums of h with respect to P_λ, respectively. Since h is monotonic on $(0,a)$ and (b,∞), and since $h(\lambda n) \to 0$ as $n \to \infty$ (by (i)), we obtain that

$$\lambda\Big(\sum_{n=1}^{[a'/\lambda]} + \sum_{n=[b'/\lambda]}^{\infty}\Big)|h(\lambda n) - \int_n^{n+1} h(\lambda t)dt|$$

$$\leq \lambda \sum_{n=1}^{[a'/\lambda]} |h(\lambda n) - h(\lambda(n+1))| + \lambda \sum_{n=[b'/\lambda]}^{\infty} |(\lambda n) - h(\lambda(n+1))|$$

$$= \lambda\Big|\sum_{n=1}^{[a'/\lambda]} (h(\lambda n) - h(\lambda(n+1)))\Big| + \lambda\Big|\sum_{n=[b'/\lambda]}^{\infty} (h(\lambda n) - h(\lambda(n+1)))\Big|$$

$$= \lambda|h(\lambda) - h(([a'/\lambda]+1)\lambda)| + \lambda|h([b'/\lambda]\lambda)|$$

$$\leq \lambda|h(\lambda)| + \lambda \max\{|h(a)|, |h(a')|\} + \lambda|h(b)| \to 0 \quad \text{as } \lambda \downarrow 0.$$

This proves (ii). (iii) follows from part (ii) directly.

(iv) It follows from (iii) that

$$\lim_{\lambda\downarrow 0} |\lambda \sum_{n=1}^{\infty} h(\lambda n) - \lambda \int_0^{\infty} h(\lambda t)dt| = 0.$$

Hence there is a $\lambda' > 0$ such that $|\lambda \sum_{n=1}^{\infty} h(\lambda n)| \leq \lambda \int_0^{\infty} |h(\lambda t)|dt + 1 = \|h\|_1 + 1$ for all $\lambda \in (0, \lambda')$. Let $\varepsilon > 0$ be arbitrary. There is an integer $N \geq 1$ such that $\|u_n - u\| < \varepsilon$ for all $n \geq N$. Thus we have for all $\lambda \in (0, \lambda')$

$$\Big\|\lambda \sum_{n=1}^{\infty} h(\lambda n)u_n - \lambda \int_0^{\infty} h(\lambda t)dt \cdot u\Big\|$$

$$\leq \Big\|\lambda \sum_{n=1}^{\infty} h(\lambda n)(u_n - u)\Big\| + \Big|\lambda \sum_{n=1}^{\infty} h(\lambda n) - \lambda \int_0^{\infty} h(\lambda t)dt\Big|\|u\|$$

$$\leq \lambda \sum_{n=1}^{N} |h(\lambda n)|\|u_n - u\| + \varepsilon\lambda \sum_{n=N+1}^{\infty} |h(\lambda n)|$$

$$+ \Big|\lambda \sum_{n=1}^{\infty} h(\lambda n) - \lambda \int_0^{\infty} h(\lambda t)dt\Big|\|u\|.$$

Since h is positive and $\lambda h(\lambda) \to 0$ as $\lambda \to 0$, with the above estimate this implies

$$\limsup_{\lambda\downarrow 0} \Big\|\lambda \sum_{n=1}^{\infty} h(\lambda n)u_n - \lambda \int_0^{\infty} h(\lambda t)dt \cdot u\Big\| \leq \varepsilon(\|h\|_1 + 1).$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\lim_{\lambda\downarrow 0} \Big\|\lambda \sum_{n=1}^{\infty} h(\lambda n)u_n - \lambda \int_0^{\infty} h(\lambda t)dt \cdot u\Big\| = 0. \qquad \blacksquare$$

LEMMA 2.2: *If a sequence* $\{u_n\}_{n=1}^{\infty} \subset X$ *converges to* $u \in X$, *then for all* $\gamma > 0$

$$\lim_{r\uparrow 1}(1-r)^{\gamma}\sum_{n=1}^{\infty} r^n n^{\gamma-1}u_n = \Gamma(\gamma)u.$$

Proof: Let $\lambda = -\ln r$ $(0 < r < 1)$. Then we have

$$\lim_{r\uparrow 1}\lambda/(1-r) = \lim_{r\uparrow 1}(-\ln r)/(1-r) = 1$$

and so

$$\begin{aligned}\lim_{r\uparrow 1}(1-r)^{\gamma}\sum_{n=1}^{\infty} r^n n^{\gamma-1}u_n &= \lim_{r\uparrow 1}\left(\frac{1-r}{\lambda}\right)^{\gamma}\lim_{r\uparrow 1}\lambda^{\gamma}\sum_{n=1}^{\infty} r^n n^{\gamma-1}u_n \\ &= \lim_{\lambda\downarrow 0}\sum_{n=1}^{\infty} e^{-\lambda n}(\lambda n)^{\gamma-1}\lambda u_n \\ &= \lim_{\lambda\downarrow 0}\lambda\int_0^{\infty} e^{-\lambda t}(\lambda t)^{\gamma-1}dt\cdot u \\ &= \int_0^{\infty} e^{-t}t^{\gamma-1}dt\cdot u = \Gamma(\gamma)u.\end{aligned}$$

Here the third equality follows by applying (iv) of Lemma 2.1 to the function $h(t) = e^{-t}t^{\gamma-1}$. ∎

PROPOSITION 2.3: *Let* $\gamma > -1$ *and* $\{x_n\}_{n=0}^{\infty}$ *be a sequence in* X. *If* $y := \lim_{n\to\infty} n^{-\gamma}\sum_{k=0}^{n-1} x_k$ *exists, then*

$$\lim_{r\uparrow 1}\frac{(1-r)^{\gamma}}{\Gamma(\gamma+1)}\sum_{n=0}^{\infty} r^n x_n = y.$$

Proof: Let $s_n := \sum_{k=0}^{n-1} x_k$ for $n \geq 1$ and $s_0 := 0$. Under the assumption: $u_n := n^{-\gamma}s_n \to y$, we have $\|s_n\| = O(n^{\gamma})$ and so $\sum r^n s_n$ converges absolutely for all $0 < r < 1$. It follows that $\sum_{n=0}^{\infty} r^n x_n = \sum_{n=0}^{\infty} r^n(s_{n+1} - s_n)$ converges absolutely for all $0 < r < 1$. Hence

$$\begin{aligned}(1-r)^{\gamma}\sum_{n=0}^{\infty} r^n x_n &= (1-r)^{\gamma}\left[\sum_{n=0}^{\infty} r^n(n+1)^{\gamma}u_{n+1} - \sum_{n=1}^{\infty} r^n n^{\gamma}u_n\right] \\ &= \frac{(1-r)^{\gamma+1}}{r}\sum_{n=1}^{\infty} r^n n^{\gamma}u_n \to \Gamma(\gamma+1)y\end{aligned}$$

as $r \to 1$, by replacing the γ in Lemma 2.2 by $\gamma + 1$. ∎

The following proposition is a continuous analog of Proposition 2.3.

PROPOSITION 2.4: *Let $\gamma > -1$ and $x : [0, \infty) \to X$ be a locally integrable function. If $y := \lim_{t\to\infty} t^{-\gamma} \int_0^t x(s)ds$ exists, then*

$$\lim_{\lambda\downarrow 0} \frac{\lambda^\gamma}{\Gamma(\gamma+1)} \int_0^\infty e^{-\lambda t} x(t)dt = y.$$

Proof: Let $v(t) := t^{-\gamma} \int_0^t x(s)ds$. For any $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that $\|v(t) - y\| < \varepsilon$ for all $t \geq t_\varepsilon$. Using integration by parts, we have

$$\begin{aligned}
&\left\| \frac{\lambda^\gamma}{\Gamma(\gamma+1)} \int_0^\infty e^{-\lambda t} x(t)dt - y \right\| \\
&\qquad = \left\| \frac{\lambda^{\gamma+1}}{\Gamma(\gamma+1)} \int_0^\infty e^{-\lambda t} \int_0^t x(s)dsdt - y \right\| \\
&\qquad = \left\| \frac{\lambda^{\gamma+1}}{\Gamma(\gamma+1)} \int_0^\infty e^{-\lambda t} t^\gamma v(t) - y \right\| \\
&\qquad \leq \frac{\lambda^{\gamma+1}}{\Gamma(\gamma+1)} \int_0^\infty e^{-\lambda t} t^\gamma \|v(t) - y\| dt \\
&\qquad \leq \frac{\lambda^{\gamma+1}}{\Gamma(\gamma+1)} \int_0^{t_\varepsilon} e^{-\lambda t} t^\gamma \|v(t) - y\| dt + \varepsilon \frac{\lambda^{\gamma+1}}{\Gamma(\gamma+1)} \int_{t_\varepsilon}^\infty e^{-\lambda t} t^\gamma dt \\
&\qquad \leq \frac{\lambda^{\gamma+1}}{\Gamma(\gamma+1)} t_\epsilon^\gamma \int_0^{t_\varepsilon} \|v(t) - y\| dt + \varepsilon \to \varepsilon \text{ as } \lambda \to 0^+.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. ∎

Applying Proposition 2.3 (resp. Proposition 2.4) to the sequence $\{x_n - y\}$ (resp. the function $x(t) - y$), we can deduce the next corollary about the convergence rates of Cesàro mean and Abel mean.

COROLLARY 2.5: *If $\|n^{-1} \sum_{k=0}^{n-1} x_k - y\| = o(n^{-\beta})$ (resp. $\|t^{-1} \int_0^t x(s)ds - y\| = o(t^{-\beta})(t \to \infty)$) for some $0 < \beta < 2$, then also $\|(1-r)\sum_{n=0}^\infty r^n x_n - y\| = o((1-r)^\beta)(r \uparrow 1)$ (resp. $\|\lambda \int_0^\infty e^{-\lambda t} x(t)dt - y\| = o(\lambda^\beta)(\lambda \downarrow 0)$.*

Remark: The converse of Proposition 2.3 (resp. Proposition 2.4) fails to hold for any $\gamma \geq 0$. For the case $\gamma > 0$, this follows from Proposition 2.8 of [10]. The following are examples for the cases: $\gamma = 1$ and $\gamma = 0$.

Example 1: Let X be a Banach space with dim $X \geq 2$. It is shown in [6] and [10, Corollary 2.4] that there exists a Cesàro-mean-bounded operator T (resp. uniformly continuous C_0-semigroup $T(\cdot)$) on X such that T^n/n (resp. $T(t)/t$) fails to converge to 0 strongly as $n \to \infty$ (resp. $t \to \infty$). Hence T (resp. $T(\cdot)$) is not Cesàro mean ergodic, i.e., there exists an $x \in X$ such that $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} T^k x$ (resp. $\lim_{t\to\infty} t^{-1} \int_0^t T(s)xds$) fails to exist (cf. Remark

(1) under Corollary 2.4 of [10]). But, if we assume that X is reflexive then, by the Abel mean ergodic theorem (cf. [6, Theorem 2.1]), $\lim_{r\uparrow 1}(1-r)\sum_{n=0}^{\infty} r^n T^n x$ (resp. $\lim_{\lambda\downarrow 0}\lambda\int_0^{\infty} e^{-\lambda t}T(t)x dt$) exists for all $x \in X$.

Example 2: Let $x_k = (-1)^k,\ k \geq 0$. Then

$$(1-r)^0 \sum_{k=0}^{\infty} r^k x_k = \sum_{k=0}^{\infty} r^k (-1)^k = \frac{1}{1+r} \to \frac{1}{2}$$

as $r \uparrow 1$. On the other hand,

$$n^0 \sum_{k=0}^{n-1} x_k = \begin{cases} 1 & (n = 2l+1), \\ 0 & (n = 2l). \end{cases}$$

Thus, $\lim_{n\to\infty} n^0 \sum_{k=0}^{n-1} x_k$ does not exist.

3. Generalized Tauberian theorems for functions and sequences of growth order $\gamma - 1$

In this section we will show that the converse of Proposition 2.4 is also true for functions $x : [0,\infty) \to X$ which satisfy $\|x(t)\| = O(t^{\gamma-1})(t \to \infty)$ (see Proposition 3.4). We start with the next lemma.

Lemma 3.1: *Let $h \in L^1((0,\infty))$ be such that* $\text{span}\{h(\lambda\cdot); \lambda > 0\}$ *is dense in $L^1((0,\infty))$, and let $f \in L^{\infty}((0,\infty), X)$. If*

$$\lim_{\lambda\downarrow 0} \lambda \int_0^{\infty} h(\lambda t) f(t) dt = 0,$$

then

$$\lim_{\lambda\downarrow 0} \lambda \int_0^{\infty} k(\lambda t) f(t) dt = 0$$

for every $k \in L^1((0,\infty))$. The assertion also holds when "$\lambda \downarrow 0$" is replaced by "$\lambda \uparrow \infty$".

Proof: Let $\varepsilon > 0$ and $k \in L^1((0,\infty))$ be arbitrary. By the assumption on h, there are positive numbers $c_1, c_2, \ldots, c_n$ and scalars $a_1, a_2, \ldots, a_n \in \mathbb{C}$ such that

$$\int_0^{\infty} \left| k(t) - \sum_{k=1}^{n} a_k h(c_k t) \right| dt \leq \varepsilon.$$

Therefore, we have for every $\lambda > 0$

$$\begin{aligned}\left\|\lambda\int_0^\infty k(\lambda t)f(t)dt\right\| \\ &= \left\|\int_0^\infty k(t)f(t/\lambda)dt\right\| \\ &\leq \left\|\int_0^\infty [k(t)-\sum_{k=1}^n a_k h(c_k t)]f(t/\lambda)dt\right\| + \left\|\int_0^\infty \sum_{k=1}^n a_k h(c_k t)f(t/\lambda)dt\right\| \\ &\leq \varepsilon\|f\|_\infty + \sum_{k=1}^n |a_k|\left\|\lambda\int_0^\infty h(c_k\lambda t)f(t)dt\right\|.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, by taking $\lambda \downarrow 0$ we get from the assumption that

$$\lim_{\lambda\downarrow 0}\lambda\int_0^\infty k(\lambda t)f(t)dt = 0.$$

This completes the proof. ∎

Lemma 3.2: *Let $h \in L^1((0,\infty))$ be such that*

$$(*)\qquad \begin{aligned}&f = 0 \quad \text{a.e. whenever } f \in L^\infty((0,\infty)) \quad \text{and} \\ &\int_0^\infty h(\lambda t)f(t)dt = 0 \quad \text{for all } \lambda > 0.\end{aligned}$$

Then span$\{h(\lambda\cdot);\lambda > 0\}$ *is dense in $L^1((0,\infty))$. In particular, the conclusion holds for the two functions: $h_1(t) := e^{-t}t^{\gamma-1},\ t \geq 0$; $h_2(t) := t^{\gamma-1}\chi_{(0,1]}(t)$, with $Re\gamma > 0$.*

Proof: The result follows from the fact $(L^1((0,\infty)))^* = L^\infty((0,\infty))$ and the Hahn-Banach theorem.

To verify condition (*) for h_1, let $f \in L^\infty((0,\infty))$ be such that

$$\int_0^\infty e^{-\lambda t}(\lambda t)^{\gamma-1}f(t)dt = 0 \quad \text{for all } \lambda > 0.$$

Then, using integration by parts we have

$$\int_0^\infty e^{-\lambda t}\int_0^t s^{\gamma-1}f(s)dsdt = 0 \quad \text{for all } \lambda > 0.$$

Since $\int_0^t s^{\gamma-1}f(s)ds$ is exponentially bounded and continuous, it follows from the uniqueness of Laplace transform that $\int_0^t s^{\gamma-1}f(s)ds = 0$ for all $t \geq 0$. This implies that $f = 0$ almost everywhere (cf. [5, Theorem II.2.9]).

To verify condition (*) for h_2, let $f \in L^\infty((0,\infty))$ be such that

$$\int_0^\infty h_2(\lambda t) f(t)dt = 0 \quad \text{for all } \lambda > 0.$$

We have for every $\lambda > 0$ $\int_0^{1/\lambda} t^{\gamma-1} f(t)dt = 0$. Hence $f = 0$ a.e. ∎

We are now in a position to prove the following generalized Tauberian theorem. The case $\gamma = 1$ is well-known (see [7], [6, Theorem 3.3]).

PROPOSITION 3.3: *Let $f \in L^\infty([0,\infty), X)$ and let $x \in X$. Then the following are equivalent:*

(a) $\lim_{\lambda\downarrow 0} \lambda^\gamma \int_0^\infty e^{-\lambda t} t^{\gamma-1} f(t)dt = \Gamma(\gamma)x$ *for some* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*;*

(a') $\lim_{\lambda\downarrow 0} \lambda^\gamma \int_0^\infty e^{-\lambda t} t^{\gamma-1} f(t)dt = \Gamma(\gamma)x$ *for all* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*;*

(b_n) $\lim_{n\to\infty} n^{-\gamma} \int_0^n s^{\gamma-1} f(s)ds = \frac{1}{\gamma}x$ *for some* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*;*

(b_t) $\lim_{t\to\infty} t^{-\gamma} \int_0^t s^{\gamma-1} f(s)ds = \frac{1}{\gamma}x$ *for some* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*;*

(b_t') $\lim_{t\to\infty} t^{-\gamma} \int_0^t s^{\gamma-1} f(s)ds = \frac{1}{\gamma}x$ *for all* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*;*

(c) $\lim_{\lambda\downarrow 0} \lambda \int_0^\infty k(\lambda t) f(t)dt = \int_0^\infty k(t)dt\ x$ *for all* $k \in L^1((0,\infty))$*.*

Moreover, if f is feebly oscillating, i.e., $\|f(s) - f(t)\| \to 0$ whenever $t \to \infty$ and $t/s \to 1$, then the above conditions are also equivalent to

(d) $\lim_{t\to\infty} f(t) = x$.

Proof: Since $\lambda^\gamma \int_0^\infty e^{-\lambda t} t^{\gamma-1} dt = \Gamma(\gamma)$ and $t^{-\gamma} \int_0^t s^{\gamma-1} ds = \frac{1}{\gamma}$ for all $\lambda, t > 0$, we may assume $x = 0$. "(a') ⇒ (a)" and "(b_t') ⇒ (b_t) ⇒ (b_n)" are obvious, and "(c) ⇒ (a') + (b_t')" follows by letting $k = h_1$ and $k = h_2$. "(a) ⇒ (c)" follows from Lemmas 3.1 and 3.2.

(b_n) ⇒ (b_t). If (b_n) holds, then $\lim_{t\to\infty} [t]^{-\gamma} \int_0^{[t]} s^{\gamma-1} f(s)ds = \frac{1}{\gamma}x$ for some $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. Since

$$\begin{aligned}
&\left| t^{-\gamma} \int_0^t s^{\gamma-1} f(s)ds - [t]^{-\gamma} \int_0^{[t]} s^{\gamma-1} f(s)ds \right| \\
&\qquad \le \left| t^{-\gamma} \int_{[t]}^t s^{\gamma-1} f(s)ds \right| + \left| (t^{-\gamma} - [t]^{-\gamma}) \int_0^{[t]} s^{\gamma-1} f(s)ds \right| \\
&\qquad \le t^{-\operatorname{Re}\gamma} \cdot t^{\operatorname{Re}\gamma - 1} \|f\|_\infty + \left| \left(\frac{t}{[t]} \right)^{-\gamma} - 1 \right| [t]^{-\operatorname{Re}\gamma} \frac{[t]^{\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \|f\|_\infty \to 0
\end{aligned}$$

as $t \to \infty$, we also have $\lim_{t\to\infty} t^{-\gamma} \int_0^t s^{\gamma-1} f(s)ds = \frac{1}{\gamma}x$.

Finally, we have for every $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$

$$\begin{aligned}
\lim_{\lambda\downarrow 0} \lambda \int_0^\infty h_2(\lambda s) f(s)ds &= \lim_{\lambda\downarrow 0} \lambda \int_0^{1/\lambda} (\lambda s)^{\gamma-1} f(s)ds \\
&= \lim_{t\to\infty} t^{-\gamma} \int_0^t s^{\gamma-1} f(s)ds.
\end{aligned}$$

Therefore "(b_t) $\Rightarrow$ (c)" follows from Lemmas 3.1 and 3.2.

"(d) $\Rightarrow$ (b_t)" can be proved easily. That (a) (with $\gamma = 1$) implies (d) for the case that f is feebly oscillating is proved in Theorem 18.3.3 of [7]. ∎

The following Tauberian theorem gives the converse of Proposition 2.4.

PROPOSITION 3.4: *Let $\gamma > -1$ and x: $[0,\infty) \to X$ be a measurable, locally integrable function. Suppose $t^{-\gamma}\int_0^t x(s)ds$ is feebly oscillating on $(0,\infty)$ and $\|\int_0^t x(s)ds\| = O(t^\gamma)(t \to \infty)$. Then*

$$\lim_{t\to\infty} t^{-\gamma}\int_0^t x(s)ds = y$$

exists if and only if

$$\lim_{\lambda\downarrow 0} \frac{\lambda^\gamma}{\Gamma(\gamma+1)}\int_0^\infty e^{-\lambda t}x(t)dt = y.$$

In particular, the conclusion holds when $\gamma > 0$ and $\|x(t)\| = O(t^{\gamma-1})(t \to \infty)$.

Proof: We may assume that $\|\int_0^t x(s)ds\| \le t^\gamma$ for all $t \ge 1$. Let $f(t) = 0$ for $0 \le t < 1$ and $f(t) = t^{-\gamma}\int_0^t x(s)ds$ for $t \ge 1$. Then $f \in L^\infty([0,\infty), X)$, and f is feebly oscillating on $[0,\infty)$.

Since

$$\lim_{t\to\infty} t^{-\gamma}\int_0^t x(s)ds = \lim_{t\to\infty} f(t),$$

and since

$$\begin{aligned}\lim_{\lambda\downarrow 0}\lambda^\gamma\int_0^\infty e^{-\lambda t}x(t)dt &= \lim_{\lambda\downarrow 0}\lambda^{\gamma+1}\int_0^\infty e^{-\lambda t}\int_0^t x(s)dsdt\\ &= \lim_{\lambda\downarrow 0}\lambda^{\gamma+1}\left(\int_0^1+\int_1^\infty\right)\left(e^{-\lambda t}\int_0^t x(s)ds\right)dt\\ &= \lim_{\lambda\downarrow 0}\lambda^{\gamma+1}\int_1^\infty e^{-\lambda t}\int_0^t x(s)dsdt \quad (\text{by } \gamma+1>0)\\ &= \lim_{\lambda\downarrow 0}\lambda^{\gamma+1}\int_0^\infty e^{-\lambda t}t^\gamma f(t)dt,\end{aligned}$$

one can deduce the conclusion from Proposition 3.3 (replacing γ by $\gamma+1$).

Here, if $\gamma > 0$ and $\|x(t)\| = O(t^{\gamma-1})(t \to \infty)$, then we may assume that $\|x(t)\| \le t^{\gamma-1}$ for all $t \ge 1$. Then for $t \ge 1$

$$\left\|\int_0^t x(s)ds\right\| \le \int_0^1 \|x(s)\|ds + \int_0^t s^{\gamma-1}ds = \int_0^1 \|x(s)\|ds + \frac{t^\gamma}{\gamma},$$

and hence $\gamma > 0$ implies that $\| \int_0^t x(s)ds\| = O(t^\gamma)(t \to \infty)$. It remains to verify that the function f, defined by $f(t) = 0$ for $0 \le t < 1$ and $f(t) = t^{-\gamma} \int_0^t x(s)ds$ for $t \ge 1$, is feebly oscillating. Indeed, we have for $1 \le t < s$

$$\begin{aligned}\|f(s) - f(t)\| &= \left\| s^{-\gamma} \int_t^s r^{\gamma-1}(r^{-(\gamma-1)}x(r))dr + (s^{-\gamma} - t^{-\gamma}) \int_0^t x(r)dr \right\| \\ &\le s^{-\gamma} \int_t^s r^{\gamma-1}dr + (t^{-\gamma} - s^{-\gamma}) \left[\int_0^1 \|x(s)\|ds + \frac{t^\gamma}{\gamma} \right] \\ &= 2\frac{1}{\gamma}\left[1 - \left(\frac{t}{s}\right)^\gamma\right] + (t^{-\gamma} - s^{-\gamma}) \int_0^1 \|x(s)\|ds,\end{aligned}$$

which tends to zero as $t \to \infty$ and $t/s \to 1$. ∎

Example 3: Let $x(t) = \begin{cases} -2, & 0 \le t \le 1; \\ t^{-3/2}, & 1 < t < \infty \end{cases}$ and $\gamma = -1/2$. Then for all $t > 0$

$$t^{-\gamma} \int_0^t x(s)ds = t^{1/2}\left[\int_0^1 (-2)ds + \int_1^t s^{-3/2}ds \right] = -2.$$

On the other hand, using L'Hospital's rule we obtain

$$\begin{aligned}&\lim_{\lambda\downarrow 0} \frac{\lambda^{-1/2}}{\Gamma(1/2)} \int_0^\infty e^{-\lambda t}x(t)dt \\ &= \lim_{\lambda\downarrow 0} \frac{2\lambda^{1/2}}{\Gamma(1/2)} \int_0^\infty (-1)e^{-\lambda t}tx(t)dt \\ &= \lim_{\lambda\downarrow 0} \frac{2\lambda^{1/2}}{\Gamma(1/2)} \left\{ 2\int_0^1 e^{-\lambda t}tdt + \int_0^1 e^{-\lambda t}t^{-1/2}dt - \int_0^\infty e^{-\lambda t}t^{-1/2}dt \right\} \\ &= \lim_{\lambda\downarrow 0} \frac{2\lambda^{1/2}}{\Gamma(1/2)}(-1)\lambda^{-1/2}\Gamma(1/2) = -2,\end{aligned}$$

which justifies the assertion of Proposition 3.4.

Next, we deduce from Proposition 3.3 its discrete analog. It is easy to see that the functions h_1 and h_2 are piecewise continuous on $(0, \infty)$ and satisfy the property that there are two numbers $b > a > 0$ such that h_1 and h_2 are monotonic on $(0, a)$ and (b, ∞). Combining Lemma 2.1 and Proposition 3.3, we obtain the following result.

Proposition 3.5: *Let $\{x_n\}_{n=0}^\infty$ be a bounded sequence in X and let $x \in X$. Then the following are equivalent:*

(a) $\lim_{r\uparrow 1}(1-r)^\gamma \sum_{n=1}^\infty n^{\gamma-1}r^n x_n = \Gamma(\gamma)x$ *for some* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*;*

(a') $\lim_{r\uparrow 1}(1-r)^\gamma \sum_{n=1}^\infty n^{\gamma-1}r^n x_n = \Gamma(\gamma)x$ *for all* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*;*

(b) $\lim_{n\to\infty} n^{-\gamma} \sum_{k=1}^n k^{\gamma-1}x_k = \frac{1}{\gamma}x$ *for some* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*;*

(b') $\lim_{n\to\infty} n^{-\gamma}\sum_{k=1}^{n} k^{\gamma-1}x_k = \frac{1}{\gamma}x$ *for all* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$;

(c) $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} x_k = x$.

Moreover, if $\{x_n\}$ *is feebly oscillating, i.e.,* $\|x_m - x_n\| \to 0$ *whenever* $n \to \infty$ *and* $n/m \to 1$*, then the above conditions are also equivalent to*

(d) $\lim_{n\to\infty} x_n = x$.

Proof: From the fact that $\lim_{\lambda\downarrow 0}\frac{\lambda}{1-e^{-\lambda}} = 1$, it is easy to see that

$$\lim_{r\uparrow 1}(1-r)^{\gamma}\sum_{n=1}^{\infty} n^{\gamma-1}r^n x_n = \lim_{\lambda\downarrow 0}\lambda^{\gamma}\sum_{n=1}^{\infty} n^{\gamma-1}e^{-\lambda n}x_n.$$

Define $f(t) := x_n$ for $n \le t < n+1$, $n = 0, 1, 2, \ldots$. Then (iii) of Lemma 2.1 implies

$$\begin{aligned}\lim_{\lambda\downarrow 0}\lambda^{\gamma}\sum_{n=1}^{\infty} n^{\gamma-1}e^{-\lambda n}x_n &= \lim_{\lambda\downarrow 0}\lambda\sum_{n=1}^{\infty} h_1(\lambda n)f(n)\\ &= \lim_{\lambda\downarrow 0}\lambda\int_0^{\infty} h_1(\lambda t)f(t)dt = \lim_{\lambda\downarrow 0}\lambda^{\gamma}\int_0^{\infty} t^{\gamma-1}e^{-\lambda t}f(t)dt\end{aligned}$$

whenever the limit on either side exists, and also

$$\begin{aligned}\lim_{n\to\infty} n^{-\gamma}\sum_{k=1}^{n} k^{\gamma-1}x_k &= \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\Big(\frac{k}{n}\Big)^{\gamma-1}f(k) = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{\infty} h_2\Big(\frac{k}{n}\Big)f(k)\\ &= \lim_{n\to\infty} n^{-1}\int_0^{\infty} h_2\Big(\frac{s}{n}\Big)f(s)ds\\ &= \lim_{n\to\infty} n^{-\gamma}\int_0^{n} s^{\gamma-1}f(s)ds\end{aligned}$$

whenever the limit on either side exists. Now the equivalence of (a)-(c) follows from the equivalence of counterparts in Proposition 3.3. "(d) $\Rightarrow$ (c)" is obvious.

To show "(c) $\Rightarrow$ (d)" for the case that $\{x_n\}$ is feebly oscillating, let $\varepsilon > 0$ be arbitrary. There exist $n_\varepsilon \in \mathbb{N}$ and $\delta = \delta_\varepsilon > 0$ such that $\|x_n - x_k\| \le \varepsilon$ for all $n_\varepsilon \le n \le k \le n(1+\delta)$. Since

$$\begin{aligned}\frac{1}{[n\delta]}\sum_{k=n+1}^{n+[n\delta]} x_k &= \frac{n+[n\delta]}{[n\delta]}\frac{1}{n+[n\delta]}\sum_{k=1}^{n+[n\delta]} x_k - \frac{n}{[n\delta]}\frac{1}{n}\sum_{k=1}^{n} x_k\\ &= \frac{n}{[n\delta]}\Bigg[\frac{1}{n+[n\delta]}\sum_{k=1}^{n+[n\delta]} x_k - \frac{1}{n}\sum_{k=1}^{n} x_k\Bigg] + \frac{1}{n+[n\delta]}\sum_{k=1}^{n+[n\delta]} x_k,\end{aligned}$$

by (c), it converges to $\frac{1}{\delta}[x-x]+x=x$ as $n\to\infty$. Using this fact we conclude that for $n \geq n_\varepsilon$

$$\begin{aligned}\|x_n - x\| &= \left\| \frac{1}{[n\delta]} \sum_{k=n+1}^{n+[n\delta]} x_k - x - \frac{1}{[n\delta]} \sum_{k=n+1}^{n+[n\delta]} (x_k - x_n) \right\| \\ &\leq \left\| \frac{1}{[n\delta]} \sum_{k=n+1}^{n+[n\delta]} x_k - x \right\| + \varepsilon,\end{aligned}$$

which tends to ε as $n\to\infty$. Since ε is arbitrary, this shows (d).

This completes the proof. ∎

Just like the derivation of Proposition 3.4 from Proposition 3.3, from Proposition 3.5 one can easily deduce the following discrete analog of Proposition 3.4. It gives the converse of Proposition 2.3.

PROPOSITION 3.6: *Suppose the sequence* $\{x_n\}_{n=0}^\infty$ *satisfies* $\|\sum_{k=0}^{n-1} x_k\| = O(n^\gamma)(n\to\infty)$ *and* $\|m^{-\gamma}\sum_{k=0}^{m-1} x_k - n^{-\gamma}\sum_{k=0}^{n-1} x_k\| \to 0$ *whenever* $n\to\infty$ *and* $n/m\to 1$, *where* $\gamma > -1$. *Then*

$$\lim_{n\to\infty} n^{-\gamma}\sum_{k=0}^{n-1} x_k = y$$

exists if and only if

$$\lim_{r\uparrow 1} \frac{(1-r)^\gamma}{\Gamma(\gamma+1)} \sum_{n=0}^{\infty} r^n x_n = y.$$

In particular, the conclusion holds when $\gamma > 0$ *and* $\|x_n\| = O(n^{\gamma-1})(n\to\infty)$.

The next proposition shows that all the limits in Proposition 3.3 become zero if f belongs to $L^q([0,\infty),X)$ with $1<q<\infty$.

PROPOSITION 3.7: *Let* $f\in L^q((0,\infty),X)$ *with* $1<q<\infty$.

(i) *If* $h\in L^p[0,\infty)$ *where* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\lim_{\lambda\downarrow 0} \lambda^{1/p}\int_0^\infty h(\lambda t)f(t)dt = \lim_{\lambda\to\infty} \lambda^{1/p}\int_0^\infty h(\lambda t)f(t)dt = 0.$$

(ii) *In particular, if* $\gamma > 1/q$, *then*

$$\lim_{\lambda\downarrow 0} \lambda^{\gamma-1/q}\int_0^\infty e^{-\lambda t}t^{\gamma-1}f(t)dt = \lim_{\lambda\to\infty} \lambda^{\gamma-1/q}\int_0^\infty e^{-\lambda t}t^{\gamma-1}f(t)dt = 0,$$

$$\lim_{t\to\infty} t^{-\gamma+1/q}\int_0^t s^{\gamma-1}f(s)ds = \lim_{t\downarrow 0} t^{-\gamma+1/q}\int_0^t s^{\gamma-1}f(s)ds = 0.$$

Proof: (i) We define for every $\lambda > 0$ a linear operator $F_\lambda : L^q((0,\infty), X) \to X$ by

$$F_\lambda(g) := \lambda^{1/p} \int_0^\infty h(\lambda t)g(t)dt \text{ for } g \in L^q[0,\infty).$$

By Hölder's inequality, we have for every $\lambda > 0$ and $g \in L^q((0,\infty), X)$

$$\begin{aligned} \|F_\lambda(g)\| &\le \lambda^{1/p}\Big(\int_0^\infty |h(\lambda t)|^p dt\Big)^{1/p}\Big(\int_0^\infty \|g(t)\|^q dt\Big)^{1/q} \\ &= \|h\|_p \|g\|_q. \end{aligned}$$

Therefore, $\|F_\lambda\| \le \|h\|_p$ for all $\lambda > 0$.

If $g \in L^q((0,\infty), X)$ is such that $g = 0$ a.e. on $[b,\infty)$ for some $b > 0$, then, since $q > 1$ implies $p < \infty$, the integrability of $|h|^p$ implies

$$\|F_\lambda(g)\| \le \Big(\int_0^{\lambda b} |h(t)|^p dt\Big)^{1/p}\Big(\int_0^b ||g(t)||^q dt\Big)^{1/q} \to 0$$

as $\lambda \downarrow 0$. Since $q < \infty$ implies the set of all $g \in L^q((0,\infty), X)$ such that $g = 0$ a.e. on $[b,\infty)$ for some $b > 0$ is dense in $L^q((0,\infty), X)$, it follows from the uniform boundedness of $\{F_\lambda; \lambda > 0\}$ that $F_\lambda(f) \to 0$ as $\lambda \downarrow 0$ for all $f \in L^q((0,\infty), X)$. This proves the equality $\lim_{\lambda\downarrow 0} \lambda^{1/p} \int_0^\infty h(\lambda t)f(t)dt = 0$.

We can interchange the roles of $h \in L^p[0,\infty)$ and $f \in L^q((0,\infty), X)$. Then a similar argument shows that

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda^{1/p} \int_0^\infty h(\lambda t)f(t)dt &= \lim_{\mu\downarrow 0} \mu^{1/q-1} \int_0^\infty h(t/\mu)f(t)dt \\ &= \lim_{\mu\downarrow 0} \mu^{1/q} \int_0^\infty h(t)f(\mu t)dt = 0. \end{aligned}$$

(ii) follows by applying (i) to functions

$$h_1(t) = e^{-t}t^{\gamma-1} \quad \text{and} \quad h_2(t) = t^{\gamma-1}\chi_{(0,1]}(t),$$

which belong to $L^p[0,\infty)$ when $p(\gamma - 1) > -1$, or $\gamma > 1/q$. ∎

4. Generalized Tauberian theorems for positive functions and sequences in Banach lattices

We first prove the following lemma which will be needed in the proof of our generalized Tauberian theorem (Proposition 4.2) for positive functions.

LEMMA 4.1: *Let Ω be a nonempty Lebesgue measurable subset of a Euclidean space $\mathbb{R}^r$, $\mathcal{B}(\Omega)$ be the σ-field of all Lebesgue measurable sets in Ω, and m be Lebesgue measure on $\mathbb{R}^r$. Let X be a Banach lattice and let W be a Banach sublattice of $L^\infty(\Omega)$ which contains all constant functions. Suppose $F : W \to X$ is a positive linear operator. Let $\{F_\alpha\}$ be a net of positive linear operators from W to X such that*

$$\lim_\alpha F_\alpha(g) = F(g), \tag{4.1}$$

for all g in a subspace D of W which contains all constant functions. If a function $f \in W$ has the property that there are two bounded sequences $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ in the closure $\overline{D}$ of D such that

$$g_n \nearrow f \quad \text{and} \quad h_n \searrow f \text{ a.e. } [m] \tag{4.2}$$

and

$$F(h_n - g_n) \searrow 0, \tag{4.3}$$

then $\lim_\alpha F_\alpha(f) = F(f)$.

Proof: Since each F_α is positive, $\|F_\alpha\| = \|F_\alpha(1)\|$. Since $1 \in D$, by assumption we have $\lim_\alpha F_\alpha(1) = F(1)$, which implies that the operators F_α are uniformly bounded. This fact implies that (4.1) holds for all g in $\overline{D}$. For the assumed function $f \in W$, we have that $g_n \nearrow f$, $h_n \searrow f$ a.e. $[m]$ and $F(h_n - g_n) \searrow 0$. Since for every $n = 1, 2, \ldots$ and for every α

$$F_\alpha(g_n) \le F_\alpha(f) \le F_\alpha(h_n),$$

we have

$$\begin{aligned} F_\alpha(g_n) - F(h_n) \le F_\alpha(g_n) - F(f) \le F_\alpha(f) - F(f) \le F_\alpha(h_n) - F(f) \\ \le F_\alpha(h_n) - F(g_n). \end{aligned}$$

Therefore we have $\|F_\alpha(f) - F(f)\| \le \|F_\alpha(g_n) - F(h_n)\| + \|F_\alpha(h_n) - F(g_n)\|$, so that

$$\begin{aligned} &\limsup_\alpha \|F_\alpha(f) - F(f)\| \\ &\le \limsup_\alpha \|F_\alpha(g_n) - F(h_n)\| + \limsup_\alpha \|F_\alpha(h_n) - F(g_n)\| \\ &\le \|F(g_n - h_n)\| + \|F(h_n - g_n)\| \\ &\to 0 + 0 \text{ as } n \to \infty. \end{aligned}$$

This shows that $\lim_\alpha F_\alpha(f) = F(f)$. ∎

Remarks: We consider the following two special cases of F in Lemma 4.1, which will be used in the proof of Proposition 4.2.

(a) Suppose $W := L^\infty(\Omega)$ and $F : W \to X$ is a positive operator defined by the following formula:

$$F(g) := \int_\Omega g d\mu \cdot z \quad \text{for } g \in L^\infty(\Omega),$$

where $z \in X$ is a positive element and μ is an m-continuous finite measure on $\mathcal{B}(\Omega)$. If (4.2) holds, then $\{h_n - g_n\}$ is a decreasing sequence of positive elements in $L^\infty(\Omega)$ such that $h_n - g_n \searrow 0$ a.e. $[m]$, then, since μ is m-continuous, $h_n - g_n \searrow 0$ a.e. $[\mu]$. By Lebesgue's dominated convergence theorem, we have $F(h_n - g_n) \searrow 0$. Hence for this F (4.2) always implies (4.3).

(b) Suppose $\Omega = [0,1]$, $U := (e^{-1}, 1]$, $W :=$ the Banach lattice consisting of all those elements $g \in L^\infty(\Omega)$ which are continuous on U. If we define $F(g) := g(1)z$ for $g \in W$, where $z \in X$ is a positive element, it is clear that (4.2) implies (4.3).

Using Lemma 4.1 and the above remark, we prove two Tauberian theorems (Propositions 4.2 and 4.4) which, like Proposition 3.4 and 3.6, give the converse of Proposition 2.4 and of Proposition 2.3, respectively. For scalar-valued functions and the case that ν is the ordinary Lebesgue measure m, they are well-known for the case $\gamma > 0$. This can be found in Widder [18, p. 203 and p. 209].

Proposition 4.2: *Let X be a Banach lattice and let ν be an m-continuous measure on $\mathcal{B}[0,\infty)$. Let x: $[0,\infty) \to X$ be a strongly measurable positive X-valued function on $[0,\infty)$ such that $\int_0^\infty e^{-\lambda t} x(t)\nu(dt)$ exists for small $\lambda > 0$ (resp. for sufficiently large λ). For $\gamma \geq 0$, if*

$$\lim_{\lambda\downarrow 0} \lambda^\gamma \int_0^\infty e^{-\lambda t} x(t)\nu(dt) \left(\textit{resp. } \lim_{\lambda\to\infty} \lambda^\gamma \int_0^\infty e^{-\lambda t} x(t)\nu(dt) \right) = z \in X_+ \textit{ exists,}$$

then

$$\lim_{t\to\infty} t^{-\gamma} \int_0^t x(s)\nu(ds) = \frac{z}{\Gamma(\gamma+1)} \left(\textit{resp. } \lim_{t\downarrow 0} t^{-\gamma} \int_0^t x(s)\nu(ds) = \frac{z}{\Gamma(\gamma+1)} \right).$$

Proof: We show only the case that $\lambda \downarrow 0$; the proof for the case $\lambda \to \infty$ is similar. Let $\lambda > 0$. First notice that since the function $e^{-\lambda t}$ is a homeomorphism from $[0,\infty]$ to $[0,1]$, we may identify $L^\infty([0,1])$ with the space $L^\infty([0,\infty])$ by the mapping $L^\infty([0,1]) \ni g(\cdot) \to g(e^{-\lambda\cdot}) \in L^\infty([0,\infty])$. Since

$0 \le |e^{-\lambda t}g(e^{-\lambda t})x(t)| \le \|g\|_\infty e^{-\lambda t}x(t)$ a.e. $[\nu]$, the assumption on x implies that the linear operator $F_\lambda\colon L^\infty([0,1]) \to X$, defined by

$$F_\lambda(g) := \lambda^\gamma \int_0^\infty e^{-\lambda s} g(e^{-\lambda s})x(s)\nu(ds) \quad \text{for } g \in L^\infty([0,1]),$$

is well-defined and is a positive linear operator such that $\|F_\lambda(g)\| \le \|g\|_\infty \cdot \|F_\lambda(1)\|$ for all $g \in L^\infty([0,1])$. Thus $F_\lambda(1) \to z$ and $\|F_\lambda\| \le \|F_\lambda(1)\| \to \|z\|$ as $\lambda \downarrow 0$. Therefore, we may assume that the operators F_λ are uniformly bounded. On the other hand, we have for every $n = 0, 1, 2, \ldots$

$$\begin{aligned} F_\lambda(t^n) &= \lambda^\gamma \int_0^\infty e^{-\lambda s}(e^{-\lambda s})^n x(s)\nu(ds) \\ &= \lambda^\gamma \int_0^\infty e^{-\lambda(n+1)s} x(s)\nu(ds) \\ &= (n+1)^{-\gamma} F_{(n+1)\lambda}(1). \end{aligned} \tag{4.4}$$

First, we assume $\gamma > 0$. Let μ be the m-continuous probability measure on $\mathcal{B}[0,1]$ defined by $\mu(A) := \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-s}\chi_A(e^{-s})s^{\gamma-1}m(ds)$ for all $A \in \mathcal{B}[0,1]$, and define the positive linear operator $F : L^\infty([0,1]) \to X$ by

$$F(g) := \int_0^1 g(t)\mu(dt) \cdot z = \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-s} g(e^{-s})s^{\gamma-1}m(ds) \cdot z \quad (g \in L^\infty([0,1])).$$

By Remark (a) after Lemma 4.1, we have that (4.2) implies (4.3). Since for all $n = 0, 1, 2, \ldots$

$$\begin{aligned} F_\lambda(t^n) &= \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-s}(e^{-s})^n s^{\gamma-1} m(ds) \cdot F_{(n+1)\lambda}(1) \\ &\to \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-s}(e^{-s})^n s^{\gamma-1} m(ds) \cdot z = F(t^n) \text{ as } \lambda \downarrow 0, \end{aligned}$$

we have $\lim_{\lambda\downarrow 0} F_\lambda(p) = F(p)$ for all polynomials $p \in L^\infty([0,1])$.

Next, consider the case $\gamma = 0$. It follows from (4.4) that $F_\lambda(t^n) \to z$ for all $n = 0, 1, 2, \ldots$. Hence

$$\lim_{\lambda\to 0^+} F_\lambda(p) = p(1)z,$$

for all polynomials $p \in L^\infty([0,1])$. Thus if we define $F(g) := g(1)z$ for all $g \in W$, where W is the Banach lattice consisting of all elements $g \in L^\infty(\Omega)$ which are continuous on $(e^{-1}, 1]$, then Remark (b) of Lemma 4.1 asserts that (4.2) implies (4.3).

We have verified that

$$\lim_{\lambda\downarrow 0} F_\lambda(g) = F(g)$$

for all $g \in D :=$ {all polynomials on $[0,1]$}. Since D is dense in $C[0,1]$, and since the function

$$f(t) := \begin{cases} 0 & \text{for } 0 \le t < e^{-1}; \\ t^{-1} & \text{for } e^{-1} \le t \le 1 \end{cases}$$

can be approximated a.e. $[m]$ on $[0,1]$ and pointwise on $(e^{-1},1]$ by increasing and decreasing sequences of continuous functions, it follows from Lemma 4.1 that

$$\begin{aligned}\lim_{\lambda\downarrow 0} F_\lambda(f) = F(f) &= \begin{cases} \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-s} f(e^{-s}) s^{\gamma-1} m(ds)\cdot z & \text{for } \gamma > 0 \\ f(1)z & \text{for } \gamma = 0 \end{cases} \\ &= \begin{cases} \frac{1}{\Gamma(\gamma)}\int_0^1 e^{-s} (e^{-s})^{-1} s^{\gamma-1} m(ds)\cdot z & \text{for } \gamma > 0 \\ z & \text{for } \gamma = 0 \end{cases} \\ &= \frac{z}{\Gamma(\gamma+1)}.\end{aligned}$$

Since

$$\begin{aligned} F_\lambda(f) &= \lambda^\gamma \int_0^\infty e^{-\lambda s} f(e^{-\lambda s}) x(s)\nu(ds) \\ &= \lambda^\gamma \int_0^{1/\lambda} x(s)\nu(ds) = t^{-\gamma}\int_0^t x(s)\nu(ds) \end{aligned} \tag{4.5}$$

with $t = 1/\lambda$, the proof is complete. ∎

Remarks: (i) The assertion of Proposition 4.2 for the case $\gamma = 0$ is also seen by the following straightforward argument. By assumption, given an $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $0 < \lambda \le \delta$ then $\| \int_0^\infty e^{-\lambda s} x(s)\nu(ds) - z\| < \epsilon$. Next there exists a sufficiently large $G(\delta) > 0$ such that

$$\left\| \int_0^t e^{-\delta s} x(s)\nu(ds) - z \right\| < \epsilon \quad \text{for all } t \ge G(\delta).$$

Since

$$z = \lim_{\mu\downarrow 0}\left(\int_0^t + \int_t^\infty\right) e^{-\mu s} x(s)\nu(ds) = \int_0^t x(s)\nu(ds) + \lim_{\mu\downarrow 0}\int_t^\infty e^{-\mu s} x(s)\nu(ds),$$

we have

$$\int_0^t e^{-\delta s} x(s)\nu(ds) \le \int_0^t x(s)\nu(ds) \le z,$$

and so

$$\left\| z - \int_0^t x(s)\nu(ds)\right\| \le \left\| z - \int_0^t e^{-\delta s} x(s)\nu(ds)\right\| < \epsilon$$

for all $t \geq G(\delta)$. This shows that $\lim_{t\to\infty} \int_0^t x(s)\nu(ds) = z$.

(ii) If $\lim_{\lambda\downarrow 0} \lambda^\gamma \int_0^\infty e^{-\lambda t}x(t)\nu(dt) = z$ exists for some $\gamma < 0$, then one must have $x = 0$ a.e. $[\nu]$. To see this let F_λ be the positive operator as defined above. Then $F_\lambda(1) \to z \in X_+$ as $\lambda \downarrow 0$. By (4.4) we have

$$0 \leq (n+1)^{-\gamma} z = \lim_{\lambda\downarrow 0}(n+1)^{-\gamma} F_{(n+1)\lambda}(1) = \lim_{\lambda\downarrow 0} F_\lambda(t^n) \leq \lim_{\lambda\downarrow 0} F_\lambda(1) = z,$$

for all $n \geq 1$. Since $\gamma < 0$, we must have that $z = 0$, so that $\|F_\lambda\| \leq \|F_\lambda(1)\| \to 0$ as $\lambda \downarrow 0$ and hence, by (4.5), $\lim_{t\to\infty} t^{-\gamma} \int_0^t x(s)\nu(ds) = 0$. Since $x(\cdot)$ is a positive X-valued function, for any $t_0 \geq 0$ we have

$$\left\| t^{-\gamma} \int_0^{t_0} x(s)\nu(ds) \right\| \leq \left\| t^{-\gamma} \int_0^{t} x(s)\nu(ds) \right\| \to 0$$

as $t \to \infty$. Since $\gamma < 0$, this implies $\int_0^t x(s)\nu(ds) = 0$ for all $t \geq 0$. It follows from [5, Corollary 2.2.7] that $x(t) = 0$ a.e. $[\nu]$.

Combining Proposition 4.2 and Proposition 2.4 we get the next corollary.

Corollary 4.3: *Let X be a Banach lattice and let x: $[0,\infty) \to X$ be a strongly measurable positive X-valued function on $[0,\infty)$ such that $\int_0^\infty e^{-\lambda t}x(t)dt$ exists for small $\lambda > 0$. Let $\gamma \geq 0$. Then*

$$\lim_{\lambda\downarrow 0} \lambda^\gamma \int_0^\infty e^{-\lambda t}x(t)dt = z$$

exists if and only if

$$\lim_{t\to\infty} t^{-\gamma} \int_0^t x(s)ds = \frac{z}{\Gamma(\gamma+1)}.$$

From Corollary 4.3, we can deduce its discrete analog as follows.

Proposition 4.4: *Let X be a Banach lattice, and $\{x_n\}_{n=0}^\infty$ be a positive X-valued sequence. Let $\gamma \geq 0$. Then*

$$z := \lim_{r\uparrow 1}(1-r)^\gamma \sum_{n=0}^\infty r^n x_n$$

exists if and only if

$$\lim_{n\to\infty} n^{-\gamma} \sum_{k=0}^{n-1} x_k = \frac{z}{\Gamma(\gamma+1)}.$$

Proof: The "if" part is contained in Proposition 2.3. We now prove the "only if" part.

Since $\lim_{r\uparrow 1}(-\ln r)/(1-r)=1$, we see, by letting $\lambda=-\ln r$, that

$$z=\lim_{\lambda\downarrow 0}\lambda^{\gamma}\sum_{n=0}^{\infty}e^{-\lambda n}x_n.$$

Define $x(t)=x_{[t]}$ for $t>0$, where $[t]$ is the largest integer less than or equal to t. Then, by Corollary 4.3, it suffices to show that

$$\lim_{\lambda\downarrow 0}\lambda^{\gamma}\int_0^{\infty}e^{-\lambda t}x(t)dt=z.$$

To see this we notice that

$$\begin{aligned}\int_0^{\infty}e^{-\lambda t}x(t)dt-\sum_{n=0}^{\infty}e^{-\lambda n}x_n&=\sum_{n=0}^{\infty}\Big(\int_0^1(e^{-\lambda t}-1)dt\Big)e^{-\lambda n}x_n\\&=\sum_{n=0}^{\infty}\eta(\lambda)e^{-\lambda n}x_n,\end{aligned}$$

where $\eta(\lambda):=\int_0^1(e^{-\lambda t}-1)dt=\frac{1}{\lambda}(1-e^{-\lambda})-1$. Since $\eta(\lambda)\uparrow 0$ as $\lambda\downarrow 0$, it follows that

$$\begin{aligned}\lambda^{\gamma}\int_0^{\infty}e^{-\lambda t}x(t)dt&=\lambda^{\gamma}\sum_{n=0}^{\infty}e^{-\lambda n}x_n+\lambda^{\gamma}\Big(\int_0^{\infty}e^{-\lambda t}x(t)dt-\sum_{n=0}^{\infty}e^{-\lambda n}x_n\Big)\\&=[1+\eta(\lambda)]\lambda^{\gamma}\sum_{n=0}^{\infty}e^{-\lambda n}x_n\to z,\end{aligned}$$

as $\lambda\downarrow 0$. This completes the proof. ∎

5. Applications to semigroups with Cesàro means of growth order α

The Cesàro means of a bounded linear operator T and a locally integrable operator semigroup $\{T(t);t>0\}$ on X are the operators

$$C_n(T):=\frac{1}{n}\sum_{k=0}^{n-1}T^k,\ n\geq 1\quad\text{and}\quad C_t:=t^{-1}\int_0^t T(s)ds,\ t>0,$$

respectively. The respective Abel means $A_r(T), 0<r<1$, and $A_\lambda, \lambda>0$, of T and $\{T(t);t>0\}$ are defined as the following.

For $0<r<1$ we define $A_r(T)x=(1-r)\sum_{n=0}^{\infty}r^nT^nx$ for $x\in D(A_r(T))$, where $D(A_r(T))$ is the set of all $x\in X$ for which the series converges. For $\lambda>0$ we define

$$A_\lambda x=\lim_{t\to\infty}\lambda\int_0^t e^{-\lambda s}T(s)xds\quad\text{for } x\in D(A_\lambda),$$

where $D(A_\lambda)$ is the set of all $x \in X$ for which the limit exists.

We first formulate the following convergence theorem.

PROPOSITION 5.1: *Let T be a bounded linear operator (resp. $T(\cdot)$ be a locally integrable operator semigroup), and let $x \in X$.*

(i) *For $\alpha > -1$, if the limit*

$$y_x := \lim_{n\to\infty} n^{-\alpha} C_n(T)x \ \ (\text{resp.} \ = \lim_{t\to\infty} t^{-\alpha} C_t x)$$

exists, then

$$\lim_{r\uparrow 1} \frac{(1-r)^\alpha}{\Gamma(\alpha+2)} A_r(T)x \left(\text{resp.} \lim_{\lambda\downarrow 0} \frac{\lambda^\alpha}{\Gamma(\alpha+2)} A_\lambda x\right) = y_x.$$

(ii) *For $\alpha = -1$, if the limit $y_x = \sum_{n=0}^\infty T^n x$ (resp. $= \int_0^\infty T(s)xds$) exists, then $\lim_{r\uparrow 1} \sum_{n=0}^\infty r^n T^n x$ (resp. $\lim_{\lambda\downarrow 0} \int_0^\infty e^{-\lambda s} T(s)xds) = y_x$; if the latter limit exists, then $(I-T)y_x = x$ (resp. $(I - T(u))y_x = \int_0^u T(s)xds$ for all $u > 0$).*

(iii) *For $-2 < \alpha < -1$, $\lim_{n\to\infty} n^{-\alpha} C_n(T)x$ (resp. $\lim_{t\to\infty} t^{-\alpha} C_t x$) exists if and only if $\lim_{r\uparrow 1} \frac{(1-r)^\alpha}{\Gamma(\alpha+2)} A_r(T)x$ (resp. $\lim_{\lambda\downarrow 0} \frac{\lambda^\alpha}{\Gamma(\alpha+2)} A_\lambda x$) exists, if and only if $x = 0$ (resp. $T(\cdot)x \equiv 0$). Hence $\frac{(1-r)^\alpha}{\Gamma(\alpha+2)} A_r(T)$ and $\{n^{-\alpha} C_n(T)\}$ (resp. $\frac{\lambda^\alpha}{\Gamma(\alpha+2)} A_\lambda$ and $t^{-\alpha} C_t$) do not converge strongly.*

(iv) *For $0 < \beta < 2$, if*

$$\|C_n(T)x - z_x\| = o(n^{-\beta}) \ (\text{resp.} \ \|C_t x - z_x\| = o(t^{-\beta})(t\to\infty)),$$

then also

$$\|A_r(T)x - z_x\| = o((1-r)^\beta)(r\uparrow 1)(\text{resp.} \ \|A_\lambda x - z_x\| = o(\lambda^\beta)(\lambda\downarrow 0)).$$

Proof: From Propositions 2.3 and 2.4, and Corollary 2.5 (with $\gamma = \alpha+1$) follow (i), the first part of (ii), the first "only if" part of (iii), and (iv). It remains to show the rest parts of (ii) and (iii).

Suppose $y_x = \lim_{r\uparrow 1} \frac{(1-r)^\alpha}{\Gamma(\alpha+2)} A_r(T)x$ exists. Let $R_r = \sum_{n=0}^\infty r^n T^n$. Then since $(1-r)^\alpha A_r(T) = (1-r)^{\alpha+1} R_r$, we have

$$\lim_{r\uparrow 1} R_r x = \begin{cases} y_x & \text{if } \alpha = -1; \\ 0 & \text{if } -2 < \alpha < -1 \end{cases},$$

so that

$$x = R_r x - rTR_r x \to \begin{cases} y_x - Ty_x & \text{if } \alpha = -1; \\ 0 - T0 = 0 & \text{if } -2 < \alpha < -1 \end{cases}$$

as $r \uparrow 1$.

Similarly, we have

$$\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda s}T(s)xds = \begin{cases} y_x & \text{if } \alpha=-1; \\ 0 & \text{if } -2<\alpha<-1 \end{cases}.$$

Then, for all $u \geq 0$, by letting $R_\lambda x = \int_0^\infty e^{-\lambda s}T(s)xds$ we have

$$\begin{aligned}\int_0^u T(s)xds &= \lim_{\lambda\downarrow 0}\int_0^u e^{-\lambda s}T(s)xds = \lim_{\lambda\downarrow 0}[R_\lambda x - e^{-\lambda u}T(u)R_\lambda x] \\ &= \begin{cases} y_x - T(u)y_x & \text{if } \alpha=-1; \\ 0-T(u)0=0 & \text{if } -2<\alpha<-1. \end{cases}\end{aligned}$$

For the case $-2<\alpha<-1$, this implies $T(\cdot)x\equiv 0$ (which is equivalent to $x=0$ in case the semigroup is nondegenerate). ∎

Remarks: (i) The above shows that for a semigroup $T(\cdot)$, if $-1<\gamma<0$ then the existence of $\lim_{t\to\infty} t^{-\gamma}\int_0^t T(s)xds = 0$ implies $T(\cdot)x\equiv 0$. As is shown in Example 3, this assertion does not hold if $T(\cdot)x$ is replaced by a general function $x(\cdot)$.

(ii) It can be shown that assertion (iv) of Proposition 5.1 still holds if we replace the small o's by big O's. Moreover, when T is power bounded (resp. $T(\cdot)=e^{\cdot A}$ is a uniformly bounded C_0-semigroup), by a different method the mean ergodic theorem with rates (cf. [1], [17, p. 293]) shows that for all $x \in N(T-I)\oplus\overline{R(T-I)}$ (resp. $N(A)\oplus\overline{R(A)}$) and $0<\beta\leq 1$

$$\begin{aligned}&\|C_n(T)x-Px\| = O(n^{-\beta})(\text{resp. } o(n^{-\beta})) \\ &\Leftrightarrow \|A_r(T)x-Px\| = O((1-r)^\beta)(\text{resp. } o((1-r)^\beta))(r\uparrow 1)\end{aligned}$$

and

$$\begin{aligned}&\|C_t x-Px\| = O(t^{-\beta})(\text{resp. } o(t^{-\beta}))(t\to\infty) \\ &\Leftrightarrow \|A_\lambda x-Px\| = O(\lambda^\beta)(\text{resp. } o(\lambda^\beta))(\lambda\downarrow 0),\end{aligned}$$

where P is the projection onto $N(A)$ along $\overline{R(A)}$.

From Propositions 3.4 and 3.6 (taking $\gamma=\alpha+1$) and Proposition 5.1 we deduce the following generalized Tauberian theorem for semigroups whose Cesàro means are of growth order α.

Proposition 5.2: *Let T be a linear operator (resp. $T(\cdot)$ be a locally integrable operator semigroup) on X. Then*

(i) *For $\alpha>-2$, suppose*

$$\|C_n(T)x\| = O(n^\alpha)(\text{resp. } \|C_t x\| = O(t^\alpha)(t\to\infty))$$

and $\{n^{-\alpha}C_n(T)x\}$ *(resp.* $t^{-\alpha}C_tx$*) is feebly oscillating, then*

$$y_x = \lim_{n\to\infty} n^{-\alpha}C_n(T)x \quad (\textit{resp.} \lim_{t\to\infty} t^{-\alpha}C_txds)$$

exists if and only if $\lim_{r\uparrow 1}\frac{(1-r)^\alpha}{\Gamma(\alpha+2)}A_r(T)x$ *(resp.* $\lim_{\lambda\downarrow 0}\frac{\lambda^\alpha}{\Gamma(\alpha+2)}A_\lambda x) = y_x$. *In case* $-2<\alpha<-1$, *these conditions are also equivalent to that* $x=0$.

(ii) *For* $\alpha \geq -1$, *suppose* $\|C_n(T)\| = O(n^\alpha)$ *(resp.* $\|C_t\| = O(t^\alpha)(t\to\infty)$*) and* $\{n^{-\alpha}C_n(T)\}$ *(resp.* $t^{-\alpha}C_t$*) is strongly feebly oscillating, then* $P := \lim_{n\to\infty} n^{-\alpha}C_n(T)$ *(resp.* $\lim_{t\to\infty} t^{-\alpha}C_tds$*) exists in the strong operator topology if and only if* $\lim_{r\uparrow 1}\frac{(1-r)^\alpha}{\Gamma(\alpha+2)}A_r(T)$ *(resp.* $\lim_{\lambda\downarrow 0}\frac{\lambda^\alpha}{\Gamma(\alpha+2)}A_\lambda) = P$ *in the strong operator topology. In case* $\alpha=-1$, *the existence of the operator* P *implies that* $I-T$ *is invertible and* $P=(I-T)^{-1}$ *(resp.* $(I-T(u))P = \int_0^u T(s)ds$ *for all* $u>0$*).*

(iii) *For* $\alpha \geq -1$, *suppose* $\|C_n(T)\| = O(n^\alpha)$ *(resp.* $\|C_t\| = O(t^\alpha)(t\to\infty)$*) and* $\{n^{-\alpha}C_n(T)\}$ *(resp.* $t^{-\alpha}C_t$*) is uniformly feebly oscillating, then* $P := \lim_{n\to\infty} n^{-\alpha}C_n(T)$ *(resp.* $\lim_{t\to\infty} t^{-\alpha}C_tds$*) exists in operator norm if and only if* $\lim_{r\uparrow 1}\frac{(1-r)^\alpha}{\Gamma(\alpha+2)}A_r(T)$ *(resp.* $\lim_{\lambda\downarrow 0}\frac{\lambda^\alpha}{\Gamma(\alpha+2)}A_\lambda) = P$ *in operator norm.*

In particular, the assertions hold when $\alpha > -1$ *and* $\|T^n\| = O(n^\alpha)$ *(resp.* $\|T(t)\| = O(t^\alpha)(t\to\infty)$*).*

Remarks: (i) Proposition 5.2 still holds if the semigroup $\{T^n\}$ (resp. $T(\cdot)$) is replaced by any sequence $\{T_n\}$ (resp. function) of operators and $C_n(T)$ is replaced by $n^{-1}\sum_{k=0}^{n-1}T_k$. So do Propositions 5.1 and 6.1.

(ii) As shown in the proof of Proposition 3.4, the condition $\|T^n\| = O(n^\alpha)$ (resp. $\|T(t)\| = O(t^\alpha)(t\to\infty)$) implies that $\{n^{-\alpha}C_n(T)\}$ (resp. $t^{-\alpha}C_t$) is bounded and strongly feebly oscillating. For the case $\alpha=0$, this means that power boundedness of T is a sufficient condition for $\{C_n(T)\}$ (resp. C_t) to be bounded and strongly feebly oscillating, and for (i) and (ii) to hold for $\alpha=0$, by Proposition 5.2. But it is not a necessary condition. Indeed, although every Cesàro-mean-ergodic positive matrix on a finite-dimensional space is necessarily power bounded (cf. [14, Chap. 1, Sec. 3], [2, p. 449]), there is an example of Cesàro-mean-ergodic operator on a Hilbert space such that $\|T^n/n\|$ does not converge to 0 [2, p. 451]. See Remark (ii) after Proposition 6.2 for an example of mean ergodic positive operator which is not power bounded but satisfies $\|T^n/n\|\to 0$. Such operators are not power bounded though $\{C_n(T)\}$, as a strongly convergent sequence, is bounded and strongly feebly oscillating.

Proposition 5.2 is concerned with general asymptotic behavior of Cesàro means and Abel means of semigroups whose Cesàro means are of growth order $O(t^\alpha)$ for $\alpha \geq -1$. In the following, we discuss particular properties for the three cases: $-1 < \alpha < 0$; $\alpha = 0$; $\alpha > 0$.

Proposition 5.3: *Assume that $-1 < \alpha < 0$. Then the following hold:*

(a) *If $y_x = \lim_{n\to\infty} n^{-\alpha} C_n(T)x$, then $Ty_x = y_x$.*

(b) *If $Tx = x (\neq 0)$, then $\lim_{n\to\infty} n^{-\alpha}\|C_n(T)x\| = \infty$. Thus*

$$\lim_{n\to\infty} n^{-\alpha} C_n(T)$$

does not exist in the strong operator topology if $\ker(T - I) \neq \{0\}$.

(c) *If $\|T^n x\| = O(\log n)$, then*

$$\lim_{n\to\infty} n^{-\alpha} C_n(T)(x - Tx) = \lim_{n\to\infty} n^{-1-\alpha}(x - T^n x) = 0.$$

(d) *If $\lim_{n\to\infty} \|T^n x\| = 0$, then for every $l \geq 1$ we have*

$$\lim_{n\to\infty} n^1 C_n(T)(x - T^l x) = x + Tx + \cdots + T^{l-1}x.$$

Proof: (a) Since $\lim_{n\to\infty} n^{1+\alpha} = \infty$, we have

$$\begin{aligned} Ty_x &= \lim_{n\to\infty} n^{-\alpha} C_n(T)Tx = \lim_{n\to\infty} \frac{1}{n^{1+\alpha}}\left(\sum_{k=0}^{n} T^k x - x\right) \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{1+\alpha} (n+1)^{-\alpha} C_{n+1}(T)x = y_x. \end{aligned}$$

Thus, if $\ker(T - I) = \{0\}$, then $y_x = 0$.

(b) We have $n^{-\alpha}\|C_n(T)x\| = \|n^{-\alpha}x\|$, and $\lim_{n\to\infty} \|n^{-\alpha}x\| = \infty$, since $\|x\| > 0$.

(d) Since

$$\begin{aligned} & n^1 C_n(T)(x - T^l x) \\ &\qquad = \sum_{k=0}^{n-1} T^k (x - T^l x) \\ &\qquad = (x + Tx + \cdots + T^{l-1}x) - (T^n x + T^{n+1}x + \cdots + T^{n+l-1}x), \end{aligned}$$

the assertion follows from the assumption.

The next is an example for the case $-1 \leq \alpha$. ∎

Example 4: Let $m \geq 1$ be an integer and N be a nilpotent operator on X with $N^{m+1} = 0, N^m \neq 0$. Let $\alpha \geq -1$ and define $T := N$. Then we have

$$\|n^{-\alpha} C_n(T)\| = n^{-\alpha-1}\left\|\sum_{k=0}^{n-1} T^k\right\| = n^{-\alpha-1}\left\|\sum_{k=0}^{m} T^k\right\| \leq \left\|\sum_{k=0}^{m} T^k\right\|$$

for all $n \geq m+1$,

$$\lim_{n\to\infty} n^{-\alpha} C_n(T) = \begin{cases} \sum_{k=0}^m T^k & \text{for } \alpha = -1; \\ 0 & \text{for } \alpha > -1, \end{cases}$$

and

$$\begin{aligned} \lim_{r\uparrow 1} (1-r)^\alpha A_r(T) &= \lim_{r\uparrow 1} (1-r)^{\alpha+1} \sum_{k=0}^{\infty} r^k T^k = \lim_{r\uparrow 1} (1-r)^{\alpha+1} \sum_{k=0}^{m} r^k T^k \\ &= \begin{cases} \sum_{k=0}^m T^k & \text{for } \alpha = -1; \\ 0 & \text{for } \alpha > -1. \end{cases} \end{aligned}$$

This justifies Proposition 5.2 for every $\alpha \geq -1$.

We now formulate the following mean ergodic theorem for Cesàro bounded semigroups, as an illustration of application of Proposition 5.2 for the case $\alpha = 0$,

PROPOSITION 5.4: *Let T be a linear operator (resp. $T(\cdot) = e^{\cdot A}$ be a C_0-semigroup of operators) on X.*

(i) *If $\{C_n(T)\}$ (resp. C_t) is bounded, then the operator P_a, defined by $P_a x := \lim_{r\uparrow 1} A_r(T)x$ (resp. $\lim_{\lambda\downarrow 0} A_\lambda x$), is a projection with range $R(P_a) = N(T-I)$ (resp. $N(A)$), null space $N(P_a) = \overline{R(T-I)}$(resp. $\overline{R(A)}$) and domain*

$$\begin{aligned} D(P_a) &= N(T-I) \oplus \overline{R(T-I)} \\ &= \{x \in X; \exists \{r_n\} \uparrow 1 \text{ s.t. w-} \lim_{n\to\infty} A_{r_n}(T)x \text{ exists}\} \end{aligned}$$

$$\text{(resp. } = N(A) \oplus \overline{R(A)} = \{x \in X; \exists \{\lambda_n\} \downarrow 0 \text{ s.t. w-} \lim_{n\to\infty} A_{\lambda_n} x \text{ exists}\}).$$

(ii) *If $\{C_n(T)\}$ (resp. C_t) is bounded and strongly feebly oscillating, the following statements hold:*

(a) *$\lim_{n\to\infty} C_n(T)x$ (resp. $\lim_{t\to\infty} C_t x$) exists if and only if*

$$\lim_{r\uparrow 1} A_r(T)x \text{ (resp. } \lim_{\lambda\downarrow 0} A_\lambda x)$$

exists, and they are equal. Thus T (resp. $T(\cdot)$) is Cesàro-mean-ergodic if and only if it is Abel-mean-ergodic.

(b) *The operator P_c, defined by*

$$P_c x := \lim_{n\to\infty} C_n(T)x \text{ (resp. } \lim_{t\to\infty} C_t x),$$

coincides with P_a. *Moreover, we have*

$$D(P_c) = \{x \in X; \exists y \in N(T-I) \text{ and } \{x_n\} \subset \mathrm{co}\{T^n x; n \geq 0\} \\ \text{s.t. w-}\lim_{n\to\infty} x_n = y\}$$

$$(\text{resp.} = \{x \in X; \exists y \in N(A) \text{ and } \{x_n\} \subset \mathrm{co}\{T(t)x; t \geq 0\} \\ \text{s.t. w-}\lim_{n\to\infty} x_n = y\}).$$

(c) *If* X *is reflexive, then* T *(resp.* $T(\cdot)$*) is Cesàro-mean-ergodic and Abel-mean-ergodic.*

Proof: (i) The boundedness of $\{C_n(T)\}$ (resp. $\{C_t; t > 0\}$) implies the boundedness of $\{A_r(T); 0 < r < 1\}$ (resp. $\{A_\lambda; \lambda > 0\}$) [10, Proposition 3.1]. Since

$$A_r(T) = (1-r)(1-rT)^{-1} = \frac{1-r}{r}\Big(\frac{1-r}{r} - (T-I)\Big)^{-1}$$

and

$$(T-I)A_r(T) = \frac{1-r}{r}A_r(T) - \frac{1-r}{r}I \to 0$$

as $r \uparrow 1$, (i) follows from the mean ergodic theorem for resolvent (cf. [19, pp. 217–218]) or from the abstract ergodic theorem (Theorem 1.1 in [15]).

(a) of (ii) follows from Proposition 5.2. (b) follows from (i), (a), and the fact that

$$x = y + (x-y) = y - \text{w-}\lim_{n\to\infty}(x_n - x) \in N(T-I) \oplus \overline{R(T-I)} = D(P_a) = D(P_c).$$

Finally, since $\{A_r(T)x; 0 < r < 1\}$ (resp. $\{A_\lambda x; \lambda > 0\}$) is bounded for all $x \in X$ and X is reflexive, (c) follows from (i) and (a). ∎

Study on asymptotic behavior of unbounded semigroups, i.e., for the case $\alpha > 0$, can be found in [9]. The next is an example for the case $\alpha > 0$.

Example 5: Let $\alpha = m \geq 1$ and define the operator $T = I + N$ and the uniformly continuous C_0-semigroup $T(t) := e^{tN} = \sum_{k=0}^{m} \frac{t^k}{k!} N^k$ for $t \geq 0$, where N is the nilpotent operator in Example 4.

We have

$$T^n = (I+N)^n = \sum_{k=0}^{m} \binom{n}{k} N^k = \sum_{k=0}^{m} \frac{n!}{k!(n-k)!} N^k,$$

and thus

$$n^{-\alpha} T^n = n^{-\alpha} \sum_{k=0}^{\alpha} \frac{n!}{k!(n-k)!} N^k \longrightarrow \frac{1}{\alpha!} N^\alpha \quad \text{as } n \to \infty.$$

Hence it follows that $\|T^n\| = O(n^\alpha)$. On the other hand, since

$$A_r(T) = (1-r)\sum_{n=0}^{\infty} r^n \sum_{k=0}^{\alpha} \binom{n}{k} N^k = (1-r)\sum_{k=0}^{\alpha}\left[\sum_{n=k}^{\infty}\binom{n}{k} r^n\right] N^k$$
$$= (1-r)\sum_{k=0}^{\alpha} r^k (1-r)^{-k-1} N^k,$$

we have

$$(1-r)^\alpha A_r(T) = r^\alpha N^\alpha + \sum_{k=0}^{\alpha-1} r^k (1-r)^{\alpha-k} N^k \longrightarrow N^\alpha \quad \text{as } r \uparrow 1,$$

in operator norm. By using the above Proposition 5.2 (iii) we find that

$$\lim_{n\to\infty} n^{-\alpha} C_n(T) = \frac{N^\alpha}{\Gamma(\alpha+2)} = \frac{N^\alpha}{(\alpha+1)!}$$

in operator norm.

Also, from the above definition of $T(\cdot)$ we see that $\|T(t)\| = O(t^\alpha)(t \to \infty)$ and $\omega_0 \le 0$, and

$$\lambda^\alpha A_\lambda = \lambda^{\alpha+1}\int_0^\infty e^{-\lambda t}\sum_{k=0}^{\alpha}\frac{t^k}{k!}N^k dt = \sum_{k=0}^{\alpha}\lambda^{\alpha-k}N^k \longrightarrow N^m \quad \text{as } \lambda \downarrow 0,$$

in operator norm. Thus by the above Proposition 5.2 (iii) we find that

$$\lim_{t\to\infty} t^{-\alpha} C_t = \frac{N^\alpha}{\Gamma(\alpha+2)} = \frac{N^\alpha}{(\alpha+1)!}$$

in operator norm.

Remark: If we choose the above operator N to be a nilpotent positive operator on a Banach lattice, then the above two examples also serve as illustrating examples of Proposition 6.1 for cases $\alpha \ge -1$ and $\alpha > 0$, respectively.

The next example shows that the assumption of being feebly oscillating is essential in Proposition 5.2(i).

Example 6: Let $T = -I + N$ with N a nilpotent operator such that $N^3 = 0$, $N^2 \ne 0$ and such that $\|N\| < 1$. It is known (see the proofs of Propositions 2.3 and 2.8 in [10]) that

$$\|C_n(T)\| \begin{cases} = O(n); \\ \ne O(n^\alpha) \quad \forall \alpha \in [0,1), \end{cases} \quad \text{and} \quad \|T^n\| \ne O(n^\alpha)$$

for all $\alpha \in [1, 2)$,

$$\liminf_{n\to\infty} \|n^{-1}C_n(T)x\| \geq \frac{1}{2}\|I-T\|^{-1}\|N^2x\|$$

for all $x \in X$, and $\|A_r(T)\| \leq 1-r$ for all $0 \leq r < 1$. It follows that $\|(1-r)A_r(T)\| \leq (1-r)^2 \to 0$ as $r \uparrow 1$ but $\{n^{-1}C_n(T)x\}$ does not converge to 0 as $n \to \infty$ if $N^2x \neq 0$. Since $\|C_n(T)\| = O(n)$, by (i) of Proposition 5.2 one can assert that $\{n^{-1}C_n(T)x\}$ is not feebly oscillating. We check this directly in the following.

Since T is clearly invertible, one can write

$$\begin{aligned} n^{-1}C_n(T) &= n^{-2}(I-T)^{-1}[n(I-T)C_n(T)] = n^{-2}(I-T)^{-1}(I-T^n) \\ &= n^{-2}(I-T)^{-1}\Big\{I - \sum_{k=0}^{2}\binom{n}{k}(-1)^{n-k}N^k\Big\} \\ &= (I-T)^{-1}\Big\{\frac{1}{n^2}[I-(-1)^n] + \frac{1}{n}(-1)^nN - (-1)^n\frac{n-1}{2n}N^2\Big\}. \end{aligned}$$

Since $\frac{n-1}{2n}N^2x \to N^2x/2 \neq 0$ and $(-1)^n$ is oscillating as $n \to \infty$, clearly $\{n^{-1}C_n(T)x\}$ is not feebly oscillating when $N^2x \neq 0$.

Since this T is not a positive operator, this example also shows that without the assumption of positivity on T the conclusion of Proposition 6.1 below may fail.

6. Applications to semigroups of positive operators

From Corollary 4.3 and Proposition 4.4 (taking $\gamma = \alpha + 1$) we deduce the following generalized Tauberian theorem for semigroups of positive operators.

PROPOSITION 6.1: *Let T be a positive operator (resp. $T(\cdot)$ be a locally integrable semigroup of positive operators) on a Banach lattice. For $\alpha \geq -1$, the following hold.*

(i) *For positive element $x \in X$,*

$$y_x = \lim_{n\to\infty} n^{-\alpha}C_n(T)x \quad \textit{(resp.} = \lim_{t\to\infty} t^{-\alpha}C_txds)$$

exists if and only if $\lim_{r\uparrow 1}\frac{(1-r)^\alpha}{\Gamma(\alpha+2)}A_r(T)x$ *(resp.* $\lim_{\lambda\downarrow 0}\frac{\lambda^\alpha}{\Gamma(\alpha+2)}A_\lambda x) = y_x$.

(ii) $P := \lim_{n\to\infty} n^{-\alpha}C_n(T)$ *(resp.* $\lim_{t\to\infty} t^{-\alpha}C_tds$*) exists in the strong operator topology if and only if* $\lim_{r\uparrow 1}\frac{(1-r)^\alpha}{\Gamma(\alpha+2)}A_r(T)$ *(resp.* $\lim_{\lambda\downarrow 0}\frac{\lambda^\alpha}{\Gamma(\alpha+2)}A_\lambda) = P$ *in the strong operator topology.*

(iii) $P := \lim_{n\to\infty} n^{-\alpha}C_n(T)$ *(resp.* $\lim_{t\to\infty} t^{-\alpha}C_t ds$*) exists in operator norm if and only if* $\lim_{r\uparrow 1} \frac{(1-r)^\alpha}{\Gamma(\alpha+2)} A_r(T)$ *(resp.* $\lim_{\lambda\downarrow 0} \frac{\lambda^\alpha}{\Gamma(\alpha+2)} A_\lambda$*)* $= P$ *in operator norm.*

In the following we present some applications of Proposition 6.1. First, using Proposition 6.1 (with $\alpha = 0$), we can prove the following mean ergodic theorem (cf. [6, Theorems 4.2 and 4.10] about (iii)) for semigroups of positive operators on Banach lattices.

PROPOSITION 6.2: *Let T be a positive operator (resp. $T(\cdot) = e^{\cdot A}$ be a C_0-semigroup of positive operators) on a Banach lattice X.*

(i) *For positive element $x \in X$, $\lim_{n\to\infty} C_n(T)x$ (resp. $\lim_{t\to\infty} C_t x$) exists if and only if $\lim_{r\uparrow 1} A_r(T)x$ (resp. $\lim_{\lambda\downarrow 0} A_\lambda x$) exists, and they are equal. Thus T (resp. $T(\cdot)$) is Cesàro-mean-ergodic if and only if T (resp. $T(\cdot)$) is Abel-mean-ergodic.*

(ii) *If T (resp. $T(\cdot)$) is Abel-mean-bounded, then the operator P, defined by*

$$Px := \lim_{n\to\infty} C_n(T)x = \lim_{r\uparrow 1} A_r(T)x \quad \text{(resp. } := \lim_{t\to\infty} C_t x = \lim_{\lambda\downarrow 0} A_\lambda x\text{)},$$

is a linear projection with range $R(P) = N(T-I)$ (resp. $N(A)$), null space $N(P) = \overline{R(T-I)}$(resp. $\overline{R(A)}$) and domain

$$\begin{aligned} D(P) &= N(T-I) \oplus \overline{R(T-I)} \\ &= \{x \in X; \exists\{r_n\} \uparrow 1 \text{ s.t. } w\text{-}\lim_{n\to\infty} A_{r_n}(T)x \text{ exists}\} \\ &= \{x \in X; \exists y \in N(T-I) \text{ and} \\ &\qquad\qquad \{n_k\} \to \infty \text{ s.t. } w\text{-}\lim_{k\to\infty} C_{n_k}(T)x = y\} \\ \text{(resp. } &= N(A) \oplus \overline{R(A)} = \{x \in X; \exists\{\lambda_n\} \downarrow 0 \text{ s.t. } w\text{-}\lim_{n\to\infty} A_{\lambda_n} x \text{ exists}\} \\ &= \{x \in X; \exists y \in N(A) \text{ and } \{t_n\} \to \infty \text{ s.t. } w\text{-}\lim_{n\to\infty} C_{t_n} x = y\}). \end{aligned}$$

(iii) *In the case that X is reflexive, the following conditions are equivalent:*
- (a) *T (resp. $T(\cdot)$) is Abel-mean-bounded;*
- (b) *T (resp. $T(\cdot)$) is Cesàro-mean-bounded;*
- (c) *T (resp. $T(\cdot)$) is Abel-mean-ergodic;*
- (d) *T (resp. $T(\cdot)$) is Cesàro-mean-ergodic.*

Proof: (i) follows from Proposition 6.1. By the same argument in the proof of (i) of Proposition 5.4, (ii) follows from (i) and the mean ergodic theorem for resolvent (cf. [19, p. 217–218]).

(iii) ((d) ⇒ (b)) is obvious, and ((b) ⇒ (a)) is well known to be true in any Banach space (cf. [20], [10, Propositions 2.1]). ((c) ⇔ (d)) is contained in (i). Finally, ((a) ⇒ (c)) follows from (ii) and the reflexivity of X.

The proof is complete. ∎

Remarks: (i) The assertion (iii) of Proposition 6.2 holds in particular when X is a Lebesgue space $L^p(\mu)$, $1 < p < \infty$. If μ is a finite measure, and if $\{T^n; n \geq 1\}$ (resp. $T(\cdot)$) is a discrete semigroup (resp. a locally integrable semigroup) of positive operators on $L^1(\mu)$ as well as on $L^\infty(\mu)$ such that

$$\sup_{0<r<1} \sup\{\|A_r(T)f\|_1/\|f\|_1; f \in L^1(\mu)\} < \infty \quad \text{and}$$
$$\sup_{0<r<1} \sup\{\|A_r(T)f\|_\infty/\|f\|_\infty; f \in L^\infty(\mu)\} < \infty$$

$$(\text{resp. } \sup_{0<\lambda<1} \sup\{\|A_\lambda f\|_1/\|f\|_1; f \in L^1(\mu)\} < \infty \quad \text{and}$$
$$\sup_{0<\lambda<1} \sup\{\|A_\lambda f\|_\infty/\|f\|_\infty; f \in L^\infty(\mu)\} < \infty),$$

then T (resp. $T(\cdot)$) is Abel-mean-ergodic on $L^1(\mu)$ (cf. [16, Lemma 3]), and hence also Cesàro-mean-ergodic on $L^1(\mu)$, by Proposition 6.1. Moreover, since now T (resp. $T(\cdot)$) is Cesàro-mean-bounded on $L^1(\mu)$ and on $L^\infty(\mu)$, in addition to the L^1-norm convergence, $\lim_{n\to\infty} C_n(T)f$ (resp. $\lim_{t\to\infty} C_t f$) exists μ-almost everywhere for all $f \in L^\infty(\mu)(\subset L^1(\mu))$ (cf. [12]), although the $f \in L^\infty(\mu)$ here cannot be replaced by $f \in L^1(\mu)$ (cf. [3]).

(ii) In [3] there is an example of positive operator T on L^p $(1 \leq p < \infty)$ such that $\sup\{\|n^{-1}\sum_{k=0}^{n-1} T^k\|; n \geq 1\} \leq 3$, $\sup\{\|T^n\|; n \geq 1\} = \infty$ (see [3], [6, p. 14]), and $\|T^n/n\| \to 0$ (cf. [2, p. 449]). By (iii) of Proposition 6.2, for $1 < p < \infty$, such T is an example of mean ergodic positive operator which is not power bounded.

(iii) Since

$$T^n/n = \frac{n+1}{n} C_{n+1}(T) - C_n(T) \quad (\text{resp. } t^{-1}T(t)\int_0^s T(u)du = \frac{t+s}{t} C_{t+s} - C_t),$$

it can be deduced from Proposition 6.2 that any Abel-mean-bounded positive operator T (resp. positive semigroup $T(\cdot)$) on a reflexive Banach lattice satisfies the property that $T^n/n \to 0$ strongly (resp. $t^{-1}T(t)\int_0^s T(u)du \to 0$ strongly as $t \to \infty$ for all $s > 0$, which is equivalent to $T(t)/t \to 0$ strongly as $t \to \infty$ in the case that $T(\cdot)$ is norm-continuous on $[0, \infty)$). The same property is satisfied by T and $T(\cdot)$ on $L^1(\mu)$ if they satisfy the condition as described in Remark (i). However, a nonpositive Cesàro-mean-bounded semigroup on a finite dimensional space may not satisfy the property that $T^n/n \to 0$ strongly (resp. $T(t)/t \to 0$ strongly as $t \to \infty$), and hence may not be Cesàro-mean-ergodic. See Corollary 2.4(i) of [10] or [6, p. 10].

(iv) ((a) $\Rightarrow$ (b)) (resp. ((c) $\Rightarrow$ (d))) also holds for positive operators (resp. operator functions) on any (not necessarily reflexive) Banach lattice (see [6]).

As remarked previously, Examples 4 and 5 (with N therein being positive) also explain Proposition 6.1 for the case $\alpha \geq -1$ and $\alpha > 0$. The next is also an example of application of Proposition 6.1 for the case $\alpha > 0$.

Example 7: Let $\alpha \geq 1$ be an integer. For $i \geq 1$, let $X_i = L^1((i-1, i])$, and $N_i\colon X_i \to X_i$ be a positive nilpotent contraction operator with $N_i^{i+1} = 0$, $N_i^i \neq 0$. Define an operator $T_i\colon X_i \to X_i$ by $T_i = I + N_i$. Then define operators $N, T\colon L^1((0,\infty)) \to L^1((0,\infty))$ by

$$Nf = \sum_{i=1}^{\infty} N_i f_i \quad \text{and} \quad Tf = \sum_{i=1}^{\infty} T_i f_i,$$

where $f_i := f \cdot \chi_{(i-1,i]} \in X_i$ for $i \geq 1$. Then N is a positive contraction on the Banach lattice $X := L^1((0,\infty))$, and $T = I + N$ is a positive operator with norm $\|T\| \leq 2$.

For $f \in D(A_r)$ we have (cf. Example 5)

$$A_r(T)f|_{(i-1,i]} = (1-r)\sum_{k=0}^{i} r^k (1-r)^{-k-1} N_i^k f_i.$$

Thus

$$(1-r)^{\alpha} A_r(T)f|_{(i-1,i]} = (1-r)^{\alpha+1}\sum_{k=0}^{i} r^k (1-r)^{-k-1} N_i^k f_i,$$

where we see that

(i) if $i = \alpha$, then $(1-r)^{\alpha+1}\sum_{k=0}^{i} r^k(1-r)^{-k-1}N_i^k f_i \to N_\alpha^\alpha f_\alpha$ as $r \uparrow 1$,

(ii) if $i < \alpha$, then $(1-r)^{\alpha+1}\sum_{k=0}^{i} r^k(1-r)^{-k-1}N_i^k f_i \to 0 = N_i^\alpha f_i$ as $r \uparrow 1$,

(iii) if $i > \alpha$, then $(1-r)^{\alpha+1}\sum_{k=0}^{i} r^k(1-r)^{-k-1}N_i^k f_i \to N_i^\alpha f_i$ as $r \uparrow 1$ if and only if $N_i^{\alpha+1} f_i = 0$.

It follows that the limit $\lim_{r\uparrow 1}(1-r)^\alpha A_r(T)f$ exists if and only if for each $i \geq 1$ the function $f_i = f|_{(i-1,i]}$ satisfies $N_i^{\alpha+1} f_i = 0$, i.e., $N^{\alpha+1}f = 0$; and in this case we have

$$\lim_{r\uparrow 1}(1-r)^\alpha A_r(T)f = N^\alpha f.$$

Then, by Proposition 6.1(i)

$$\lim_{n\to\infty} n^{-\alpha} C_n(T)f = \frac{1}{\Gamma(\alpha+2)} N^\alpha f = \frac{N^\alpha f}{(\alpha+1)!}.$$

To consider the continuous case, let $T(t) := e^{tN} = \sum_{k=0}^{\infty}(t^k/k!)N^k$ for $t \geq 0$, where N is the one defined in the last paragraph. Thus $\{T(t)\}$ is a C_0-semigroup of positive operators on the Banach lattice X with generator N.

For $\lambda > 0$ and $f \in D(A_\lambda)$, writing $f = \sum_{i=1}^{\infty} f_i$ with $f_i \in X_i$ for all $i \geq 1$, we have

$$A_\lambda f|_{(i-1,i]} = \lambda \int_0^\infty e^{-\lambda t} \sum_{k=0}^{i} \frac{t^k}{k!} N_i^k f_i dt = \sum_{k=0}^{i} \lambda^{-k} N_i^k f_i,$$

where we see that

(i) if $i = \alpha$, then $\lim_{\lambda\downarrow 0} \lambda^\alpha \sum_{k=0}^{i} \lambda^{-k} N_i^k f_i = N_\alpha^\alpha f_\alpha$,

(ii) if $i < \alpha$, then $\lim_{\lambda\downarrow 0} \lambda^\alpha \sum_{k=0}^{i} \lambda^{-k} N_i^k f_i = 0 = N_i^\alpha f_i$,

(iii) if $i > \alpha$, then $\lim_{\lambda\downarrow 0} \lambda^\alpha \sum_{k=0}^{i} \lambda^{-k} N_i^k f_i = N_i^\alpha f_i$ if and only if $N_i^{\alpha+1} f_i = 0$.

Thus, as in the discrete case, we see that the limit $\lim_{\lambda\downarrow 0} \lambda^\alpha A_\lambda f$ exists if and only if $N^{\alpha+1} f = 0$; and in this case we have $\lim_{\lambda\downarrow 0} \lambda^\alpha A_\lambda f = N^\alpha f$, then, by Proposition 6.1(i)

$$\lim_{t\to\infty} t^{-\alpha} C_t f = \frac{N^\alpha f}{\Gamma(\alpha+2)} = \frac{N^\alpha f}{(\alpha+1)!}.$$

In order to get a brief view of the behaviour of the sequence $\{n^{-\alpha} C_n(T)x\}$ with $-1 \leq \alpha < 0$, we give the following examples.

Example 8: Let $0 < \beta < 1$, and let μ be the measure on $\mathbb{N} = \{1, 2, \ldots\}$ defined by $\mu(\{n\}) = n^{-\beta}$, $n \geq 1$. Define T: $L^1(\mu) \to L^1(\mu)$ by $Tf(1) = 0$ and $Tf(n) = f(n-1)$ for $n \geq 2$. Thus, T is a positive linear operator on $L^1(\mu)$. If $j \geq 1$, then put

$$S_n(j) := \left\| \sum_{k=0}^{n-1} T^k \chi_{\{j\}} \right\|_1 = \|\chi_{\{j,j+1,\ldots,j+n-1\}}\|_1 = \sum_{k=j}^{j+n-1} k^{-\beta}.$$

It follows that $S_n(j) \leq n \cdot j^{-\beta}$, and that

$$S_n(j) \leq j^{-\beta} + \int_j^{j+n-1} t^{-\beta} dt = j^{-\beta} + \frac{(j+n-1)^{1-\beta} - j^{1-\beta}}{1-\beta},$$

and

$$S_n(j) \geq \int_j^{j+n} t^{-\beta} dt = \frac{(j+n)^{1-\beta} - j^{1-\beta}}{1-\beta}.$$

Hence

$$\frac{(j+n)^{1-\beta} - j^{1-\beta}}{1-\beta} \leq S_n(j) \leq 1 + \frac{(j+n)^{1-\beta} - j^{1-\beta}}{1-\beta}, \tag{6.1}$$

and furthermore

$$\frac{j}{1-\beta}((1+(n/j))^{1-\beta}-1) \le \frac{\|\chi_{\{j,j+1,\dots,j+n-1\}}\|_1}{\|\chi_{\{j\}}\|_1} \le n. \tag{6.2}$$

Here we use the elementary fact that

$$\frac{(1+t)^{1-\beta}-1}{t} \le 1-\beta \quad (t>0) \quad \text{and} \quad \lim_{t\downarrow 0} \frac{(1+t)^{1-\beta}-1}{t} = 1-\beta. \tag{6.3}$$

By this, given $\widetilde{\beta}$ with $\beta < \widetilde{\beta} < 1$, there exists $\delta(\widetilde{\beta}) > 0$ so that $0 < t < \delta(\widetilde{\beta})$ implies $(1+t)^{1-\beta} - 1 > (1-\widetilde{\beta})t$. Then, by (6.2), $0 < n/j < \delta(\widetilde{\beta})$ implies

$$\frac{1-\widetilde{\beta}}{1-\beta} n \le \frac{\|\sum_{k=0}^{n-1} T^k \chi_{\{j\}}\|_1}{\|\chi_{\{j\}}\|_1} \le n. \tag{6.4}$$

Proposition 6.3: *The positive linear operator T as defined above on L^1 satisfies the following properties:*

(i) $\|T^n\| = 1$ *for all* $n \ge 1$;

(ii) $\lim_{n\to\infty} \|T^n f\|_1 = 0$ *for all* $f \in L^1$, *and* $(T-I)L^1(\mu)$ *is dense in* $L^1(\mu)$;

(iii) *The set* $M_{-1} = \{f \in L^1 : \lim_{n\to\infty} \sum_{k=0}^{n} T^k f \text{ exists}\}$ *is a dense subspace of* L^1, *and* $M_{-1}^+ = \{f \in M_{-1}; f \ge 0\} = \{0\}$.

(iv) *If* $-1 < \alpha \le -\beta$, *then*

$$M_\alpha^+ := \{0 \le f \in L^1 : \lim_{n\to\infty} n^{-\alpha} C_n(T) f \text{ exists}\} = \{0\},$$

and M_{-1} *is a proper subset of the set*

$$M_\alpha = \{f \in L^1 : \lim_{n\to\infty} n^{-\alpha} C_n(T) f \text{ exists}\},$$

(v) *If* $-\beta < \alpha < 0$, *then the set* M_α^+ *is a dense subset of* $\{f \in L^1 : f \ge 0\}$, *but* $M_\alpha^+ \ne \{f \in L^1 : f \ge 0\}$.

Proof: (i) Let $x_m := m^\beta \chi_{\{m\}}$. Since $\|x_m\|_1 = 1$ and $\|T^n x_m\|_1 = (\frac{m}{n+m})^\beta \to 1$ as $m \to \infty$, we have $\|T^n\| = 1$ for all $n \ge 1$.

(ii) Since the definition of T implies that for every $k \in \mathbb{N}$ $\|T^n \chi_{\{k\}}\|_1 = \|\chi_{\{n+k\}}\|_1 = (n+k)^{-\beta} \to 0$ as $n \to \infty$, it follows that

$$\|Pf\|_1 = \|\lim_{n\to\infty} T^n f\|_1 = 0$$

for all $f \in L^1(\mu)$. In particular, $T^n(T-I) \to 0$ strongly. Then it follows from the mean ergodic theorem that $L^1(\mu) = N(P) = \overline{R(T-I)}$.

(iii) Since the functions $f = (T-I)g$ are dense by (ii) and satisfy $\sum_{k=0}^{n} T^k f = T^{n+1}g - g \to -g$ as $n \to \infty$, also by (ii), we see that M_{-1} is a dense subspace of $L^1(\mu)$. Next, if $0 \le f \in L^1(\mu)$ and $f \ne 0$, then $f(j) > 0$ for some $j \ge 1$ so that $\lim_{n\to\infty} \|\sum_{k=0}^{n} T^k f\|_1 \ge \lim_{n\to\infty} f(j)S_{n+1}(j) = \infty$, by (6.1). Therefore $M_{-1}^{+} = \{0\}$.

(iv) Let $-1 < \alpha \le -\beta$. It follows from (6.1) that

$$\|n^{-\alpha}C_n(T)\chi_{\{j\}}\|_1 = \frac{1}{n^{1+\alpha}}\Big\|\sum_{k=0}^{n-1} T^k\chi_{\{j\}}\Big\|_1 \ge \frac{1}{n^{1+\alpha}}\cdot\frac{(j+n)^{1-\beta} - j^{1-\beta}}{1-\beta},$$

and hence, by $0 < 1+\alpha \le 1-\beta$, we have

$$\limsup_{n\to\infty} \|n^{-\alpha}C_n(T)\chi_{\{j\}}\|_1 \ge \frac{1}{1-\beta}.$$

Since $\ker(T-I) = \{0\}$, it then follows from Proposition 5.3(a) that $\chi_{\{j\}} \notin M_\alpha$. This shows that

$$\{0 \le f \in L^1(\mu) : \lim_{n\to\infty} n^{-\alpha}C_n(T)f \text{ exists}\} = \{0\}.$$

We next prove that M_{-1} is a proper subspace of M_α. To do this it suffices to show the existence of a function f in M_α with $\lim_{n\to\infty} \|\sum_{k=0}^{n} T^k f\|_1 = \infty$.

Let $k_1 = 1$, and $l_1 > k_1$ be the smallest integer satisfying

$$\|\chi_{\{k_1, k_1+1, \ldots, k_1+l_1-1\}}\|_1 \ge 1.$$

Suppose $k_1 < l_1 < \cdots < k_{n-1} < l_{n-1}$ has been determined. Then there exists $\widetilde{k_n} > l_{n-1}$ such that $b \ge \widetilde{k_n}$ implies

$$\frac{1}{b^{1+\alpha}}\Big\|\sum_{m=0}^{b-1} T^m\{(\chi_{\{k_1\}} - \chi_{\{l_1\}}) + \cdots + (\chi_{\{k_{n-1}\}} - \chi_{\{l_{n-1}\}})\}\Big\|_1 \le 2^{-n}. \tag{6.5}$$

Next we can take a sufficiently large integer $d_n > \widetilde{k_n}$, with $\|\chi_{\{d_n\}}\|_1 < 2^{-n}$, so that

$$\|m^{-\alpha}C_m(T)\chi_{\{d_n\}}\|_1 < 2^{-n} \quad \text{for all } m, \text{ with } 1 \le m \le \widetilde{k_n}. \tag{6.6}$$

Let $k_n = d_n$, and $l_n > k_n$ be the smallest integer satisfying

$$\|\chi_{\{k_n, k_n+1, \ldots, k_n+l_n-1\}}\|_1 \ge 1. \tag{6.7}$$

Continuing this process we can determine two strictly increasing sequences $\{k_n\}$ and $\{l_n\}$ of positive integers, with $k_n < l_n$ for all $n \ge 1$. Then the function

$$f = \sum_{n=1}^{\infty}(\chi_{\{k_n\}} - \chi_{\{l_n\}})$$

is in $L^1(\mu)$, and satisfies, by (6.5) and (6.6), and (6.7), that

$$\lim_{m\to\infty} \|m^{-\alpha}C_m(T)f\|_1 = 0, \quad \text{and} \quad \lim_{m\to\infty} \|\sum_{n=0}^{m} T^n f\|_1 = \infty.$$

Thus $f \in M_\alpha \setminus M_{-1}$.

(v) Let $-\beta < \alpha < 0$. Since (6.1) implies

$$\|n^{-\alpha}C_n(T)\chi_{\{j\}}\|_1 \le \frac{1}{n^{1+\alpha}}\Big\{1 + \frac{(j+n)^{1-\beta} - j^{1-\beta}}{1-\beta}\Big\},$$

it follows from $0 < 1-\beta < 1+\alpha$ that

$$\lim_{n\to\infty} \frac{1}{n^{1+\alpha}}\Big\{1 + \frac{(j+n)^{1-\beta} - j^{1-\beta}}{1-\beta}\Big\} = 0.$$

Thus $\chi_{\{j\}} \in M_\alpha^+$, where $M_\alpha^+ = \{f \in M_\alpha : f \ge 0\}$. This shows that M_α^+ is a dense subset of $\{f \in L^1(\mu) : f \ge 0\}$.

We lastly prove that M_α^+ is a proper subset of $\{f \in L^1(\mu) : f \ge 0\}$. To do this, we note by (6.4) that for each $n \ge 1$ there correspond two positive integers s_n and t_n, with $0 < t_n/s_n < \delta(\widetilde{\beta})$, so that

$$\frac{\| \sum_{k=0}^{t_n-1} T^k \chi_{\{s_n\}}\|_1}{\|\chi_{\{s_n\}}\|_1} > \frac{1-\widetilde{\beta}}{1-\beta} t_n > 2^n t_n^{1+\alpha}, \tag{6.8}$$

where the last inequality holds when t_n is chosen so largely that the inequality $(1-\widetilde{\beta}) > (1-\beta)2^n t_n^\alpha$ is true. Here we may assume that $\{s_n\}$ and $\{t_n\}$ are strictly increasing sequences. Let w_n be a positive real number satisfying $\|w_n\chi_{\{s_n\}}\|_1 = 2^{-n}$. Then the function

$$f = \sum_{n=1}^{\infty} w_n \chi_{\{s_n\}}$$

is a positive function in $L^1(\mu)$, and satisfies, by (6.8), that

$$\begin{aligned}\|t_n^{-\alpha}C_{t_n}(T)f\|_1 &\ge \frac{1}{t_n^{1+\alpha}} \Big\| \sum_{k=0}^{t_n-1} T^k(w_n\chi_{\{s_n\}})\Big\|_1 \\ &\ge \frac{1}{t_n^{1+\alpha}} \|w_n\chi_{\{s_n\}}\|_1 \frac{\| \sum_{k=0}^{t_n-1} T^k \chi_{\{s_n\}}\|_1}{\|\chi_{\{s_n\}}\|_1} \\ &\ge \frac{1}{t_n^{1+\alpha}} \cdot 2^{-n} \cdot 2^n t_n^{1+\alpha} = 1 \quad (n \ge 1).\end{aligned}$$

Hence $\limsup_{n\to\infty} \|n^{-\alpha}C_n(T)f\|_1 \ge 1$, and this implies, by Proposition 5.3(a) and the fact $\ker(T-I) = \{0\}$, that $f \notin M_\alpha$. The proof is complete. ∎

Remark: The above proof of $M_\alpha^+ \neq \{f \in L^1(\mu) : f \geq 0\}$ could be replaced by the following argument: By (ii), $I - T$ is not invertible (but its range is dense). Hence, by Theorem 2.17 and Proposition 2.1(iii) of Derriennic and Lin [4], $M_\alpha \neq L^1(\mu)$. Consequently, $M_\alpha^+ \neq \{f \in L^1(\mu) : f \geq 0\}$. This argument was communicated to the authors by the referee.

Example 9: For $-1 \leq \alpha < 0$ put $\beta = -\alpha$, and let

$$d_1 = 1, \quad \text{and} \quad 1 - d_2 - \cdots - d_n = 1/n^\beta \text{ for } n \geq 2.$$

Thus,

$$d_n = \frac{1}{(n-1)^\beta} - \frac{1}{n^\beta} = \frac{n^\beta - (n-1)^\beta}{\{n(n-1)\}^\beta},$$

and so

$$\frac{\beta}{n^{1+\beta}} = \frac{\beta n^{\beta-1}}{n^{2\beta}} \leq d_n \leq \frac{\beta(n-1)^{\beta-1}}{(n-1)^{2\beta}} = \frac{\beta}{(n-1)^{1+\beta}}. \tag{6.9}$$

Define a measure μ on $\mathbb{N}$ by $\mu(\{n\}) = d_n$ for $n \in \mathbb{N}$. It follows that $\mu(\mathbb{N}) < \infty$. Then define a positive linear operator T on $L^1(\mu)$ by

$$Tf(n) = \begin{cases} \sum_{k=1}^\infty f(k)\mu(\{k\}) & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ f(n-1) & \text{if } n \geq 3. \end{cases} \tag{6.10}$$

Proposition 6.4: *Let $-1 \leq \alpha < 0$. The positive linear operator T on L^1 defined above satisfies the following properties:*

(i) $\|T^n\| = n + 1$ *and* $\|n^{-1}\sum_{k=0}^{n-1} T^k\| = 2^{-1}(n+1)$ *for all* $n \geq 1$, *and* $\ker(T - I) = \{c\chi_{\{1\}} : c \in \mathbb{R}\}$;

(ii) $\{0 \leq f \in L^1 : \lim_{n\to\infty} n^{-\alpha}C_n(T)f \text{ exists}\} = \{0\}$;

(iii) *the set* $M_{-1} = \{f \in L^1 : \lim_{n\to\infty} n^1 C_n(T)f \text{ exists}\}$ *is a dense subspace of* L^1;

(iv) *If* $-1 < \alpha < 0$, *then there exists* $f \in L^1$ *such that* $y_f = \lim_{n\to\infty} n^{-\alpha}C_n(T)f$ *exists and* $y_f \neq 0$.

Proof: (i) It is clear from the definition of T that $\ker(T-1) = \{c\chi_{\{1\}} : c \in \mathbb{R}\}$. Next, we show that $\|T^n\| = n+1$ for all $n \geq 1$. Since $k \geq 2$ implies $\|T^n\chi_{\{k\}}\|_1 = \mu(\{k, k+1, \ldots, k+n\}) = \sum_{j=0}^n d_{k+j}$, we have

$$\|T^n\| \geq \sup_{k\geq 2} \frac{1}{d_k}\sum_{j=0}^n d_{k+j}.$$

Here we notice that $1 \geq d_{k+j}/d_k \geq d_{k+n}/d_k$ for $0 \leq j \leq n$, and that

$$\lim_{k\to\infty} d_{k+n}/d_k = \lim_{k\to\infty} \frac{\{(k+n)^\beta - (k+n-1)^\beta\}}{\{(k+n)(k+n-1)\}^\beta} \cdot \frac{\{k(k-1)\}^\beta}{\{k^\beta - (k-1)^\beta\}} = 1.$$

Hence $\|T^n\| \geq n+1$. On the other hand, it is clear by the definition of T that $\|T^n f\|_1 \leq (n+1)\|f\|_1$. Therefore, we conclude that $\|T^n\| = n+1$. By a similar argument, we see that

$$\left\| n^{-1} \sum_{k=0}^{n-1} T^k \right\| = \frac{1}{n} \sum_{k=1}^{n} k = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Hence T is not mean ergodic.

(ii) Let $0 \leq f \in L^1(\mu)$ and $\|f\|_1 > 0$. Then $\lim_{n\to\infty} \|C_n(T)f\|_1 > 0$ exists (but it may be ∞). In fact, by the definition of T we see that $\|(T^n f)\chi_{\mathbb{N}\setminus\{1\}}\|_1 \downarrow 0$ as $n \uparrow \infty$, and that $\{T^n f(1)\}_{n=0}^\infty$ is a positive increasing sequence, so that $\lim_{n\to\infty} T^n f(1)$ exists (but it may be ∞). By using these the existence of $\lim_{n\to\infty} \|C_n(T)f\|_1 \in (0,\infty]$ follows. Hence $\lim_{n\to\infty} \|n^{-\alpha} C_n(T)f\|_1 = \infty$ for all $-1 \leq \alpha < 0$. Thus, for every α with $-1 \leq \alpha < 0$,

$$\{0 \leq f \in L^1(\mu) : \lim_{n\to\infty} n^{-\alpha} C_n(T)f \text{ exists}\} = \{0\}.$$

(iii) Let $N \geq 2$, and put

$$f = (1 - d_2 - \cdots - d_{N-1})\chi_{\{1\}} - \chi_{\{N\}}. \tag{6.11}$$

Then

$$T^k f(1) = 1 - d_2 - \cdots - d_{N+k-1} = \frac{1}{(N+k-1)^\beta},$$

and thus

$$\sum_{k=0}^{n-1} T^k f(1) = \sum_{l=N-1}^{N+n-2} \frac{1}{l^\beta}. \tag{6.12}$$

Next, let

$$g = (1 - d_2 - \cdots - d_{K-1})\chi_{\{1\}} - \chi_{\{K\}}, \tag{6.13}$$

where we assume that $N < K$. Then (6.12) implies

$$\sum_{k=0}^{n-1} T^k (f-g)(1) = \sum_{l=N-1}^{N+n-2} \frac{1}{l^\beta} - \sum_{l=K-1}^{K+n-2} \frac{1}{l^\beta}.$$

By this,

$$\lim_{n\to\infty} n^1 C_n(T)(f-g) = \lim_{n\to\infty} \sum_{k=0}^{n-1} T^k(f-g) = \Big(\sum_{l=N-1}^{K-2} \frac{1}{l^\beta}\Big)\chi_{\{1\}} - \sum_{j=N}^{K-1} \chi_{\{j\}}$$

in $L^1(\mu)$, i.e., $f-g \in M_{-1}$. Furthermore, by (6.11) and (6.13),

$$\begin{aligned} f-g &= ((N-1)^{-\beta}\chi_{\{1\}} - \chi_{\{N\}}) - ((K-1)^{-\beta}\chi_{\{1\}} - \chi_{\{K\}}) \\ &= \{\frac{1}{(N-1)^\beta} - \frac{1}{(K-1)^\beta}\}\chi_{\{1\}} - \chi_{\{N\}} + \chi_{\{K\}}, \end{aligned}$$

where $\mu(\{n\}) = d_n \sim \beta/n^{1+\beta} \to 0$ as $n \to \infty$, by (6.9). This shows that $\chi_{\{1\}}$ can be approximated in $L^1(\mu)$ by the functions in M_{-1} of the form of multiples of $f-g$, where f and g are defined by (6.11) and (6.13), respectively. Hence $\chi_{\{1\}} \in \overline{M_{-1}}$. Then, using the relation

$$\chi_{\{N\}} = -(f-g) + \{(N-1)^{-\beta} - (K-1)^{-\beta}\}\chi_{\{1\}} + \chi_{\{K\}},$$

and letting $K \to \infty$, we see that $\chi_{\{N\}} \in \overline{M_{-1}}$ for all $N \geq 2$.

(iv) Suppose $-1 < \alpha < 0$. Let $e_n := \sum_{l=N-1}^{N+n-2} \frac{1}{l^\beta} / \int_{N-1}^{N+n-1} x^{-\beta} dx$. Since $0 < \beta = -\alpha < 1$, it is easy to see that $e_n \to 1$ as $n \to \infty$. It follows that the function f defined in (6.11) satisfies

$$\begin{aligned} \lim_{n\to\infty} n^{-\alpha} C_n(T) f(1) &= \lim_{n\to\infty} \frac{1}{n^{1-\beta}} \sum_{k=0}^{n-1} T^k f(1) \\ &= \lim_{n\to\infty} e_n \frac{\int_{N-1}^{N+n-1} x^{-\beta} dx}{n^{1-\beta}} \\ &= \lim_{n\to\infty} \frac{(N+n-1)^{-\beta}}{(1-\beta)n^{-\beta}} = \frac{1}{1-\beta}. \end{aligned}$$

Since

$$\lim_{n\to\infty} \int_{\mathbb{N}\setminus\{1\}} \sum_{k=0}^{n-1} |T^k f| d\mu = \mu([N,\infty)) < \infty,$$

we get

$$y_f = \lim_{n\to\infty} n^{-\alpha} C_n(T) f = \frac{1}{1-\beta}\chi_{\{1\}} \quad (\text{in } L^1(\mu)).$$

This completes the proof of Proposition 6.4. ∎

Remarks: (i) On the other hand, if $f = \chi_{\{n\}}$ for some $n \in \mathbb{N}$, then we have, by $\mu(\mathbb{N}) = 2 < \infty$, that

$$\lim_{m\to\infty} \|T^m f - \mu(\{n, n+1, \ldots\})\chi_{\{1\}}\|_1 = 0$$

and so

$$\lim_{m\to\infty}\left\|\frac{1}{m}\sum_{k=0}^{m-1}T^kf-\mu(\{n,n+1,\dots\})\chi_{\{1\}}\right\|_1=0.$$

Therefore, $M_0 := \{f \in L^1(\mu) : \lim_{n\to\infty} C_n(T)f \text{ exists}\}$ is a dense subspace of $L^1(\mu)$, and so is the set $M_\epsilon := \{f \in L^1(\mu) : \lim_{n\to\infty} n^{-\epsilon}C_n(T)f \text{ exists}\}$ for all $\epsilon \geq 0$. (Notice that from (iii) it follows actually that this is true for all $\epsilon \geq -1$)

But, now, let $0 < \epsilon < 1$. Then

$$\left\|n^{-1-\epsilon}\sum_{k=0}^{n-1}T^k\right\|=n^{-\epsilon}(n+1)2^{-1}\to\infty \quad (n\to\infty),$$

so that there exists a function $f \in L^1(\mu)$, with $f \geq 0$ and $\|f\|_1 > 0$, such that $\lim_{n\to\infty}\frac{1}{n^{1+\epsilon}}\sum_{k=0}^{n-1}T^kf$ does not exist. Lastly, since $\|n^{-2}\sum_{k=0}^{n-1}T^k\| = \frac{n+1}{2}n \leq 1$, we see from the above-mentioned result that $\lim_{n\to\infty}\|n^{-2}\sum_{k=0}^{n-1}T^kf\|_1 = 0$ for all $0 \leq f \in L^1(\mu)$.

(ii) As shown in (i) and (iii) of Proposition 6.4, Example 9 exhibits a positive operator T on L^1 with $\|T^n\| = n+1$ and $\|n^{-1}\sum_{k=0}^{n-1}T^k\| = 2^{-1}(n+1)$ for all $n \geq 1$, such that $\lim_{n\to\infty}\sum_{k=0}^{n}T^kf$ exists (in particular, $\lim_{n\to\infty}\|T^nf\|_1 = 0$) for f in a dense subset of L^1. In connection with this example, it is interesting to note here that Kornfeld and Kosek [8] constructed, for any $\delta \in (0,1)$, a mean ergodic positive operator T on L^1 with $\lim_{n\to\infty}\|T^n\|/n^{1-\delta} = \infty$, and showed that the Cesàro-mean-boundedness of a positive L^1 operator T implies $\|T^n\| = o(n^{1-\epsilon})$ for some $\epsilon > 0$.

Acknowledgement: The authors are grateful to the referee for his careful reading and valuable suggestions.

References

[1] P. L. Butzer and A. Gessinger, *Ergodic theorems for semigroups and cosine operator functions at zero and infinity with rates; applications to partial differential equations. A survey,* Contemporary Mathematics **190** (1995), 67–93.

[2] Y. Derriennic, *On the mean ergodic theorem for Cesàro bounded operators,* Colloquium Mathematicum **84/85** (2000), 443–355.

[3] Y. Derriennic and M. Lin, *On invariant measures and ergodic theorems for positive operators,* Journal of Functional Analysis **13** (1973), 252–267.

[4] Y. Derriennic and M. Lin, *Fractional Poisson equations and ergodic theorems for fractional coboundaries,* Israel Journal of Mathematics **123** (2001), 93–130.

[5] J. Diestel and J.J. UHL, JR., *Vector Measures,* Mathematical Surveys, No. 15, American Mathematical Society, Providence, RI, 1979.

[6] R. Emilion, *Mean-bounded operators and mean ergodic theorems,* Journal of Functional Analysis **61** (1985), 1–14.

[7] E. Hille and R.S. Phillips, *Functional Analysis and Semigroups,* American Mathematical Society Colloq. Publ., vol. 31, American Mathematical Society, Providence, RI, 1957.

[8] I. Kornfeld and W. Kosek, *Positive L^1 operators associated with nonsingular mappings and an example of E. Hille,* Colloquium Mathematicum **98** (2003), 63–77.

[9] Y.-C. Li and S.-Y. Shaw, *Mean ergodicity and mean stability of regularized solution families,* Mediterranian Journal of Mathematics **1** (2004), 175–193.

[10] Y.-C. Li, R. Sato and S.-Y. Shaw, *Boundedness and growth order of means of discrete and continuous semigroups of operators,* submitted.

[11] M. Maxwell and M. Woodroofe, *Central limit theorems for additive functionals of Markov chains,* Annals Probability **28** (2000), 713–724.

[12] R. Sato, *Positive operators and the ergodic theorem,* Pacific Journal of Mathematics **76** (1978), 215–219.

[13] R. Sato, *Ratio limit theorems and applications to ergodic theory,* Studia Mathematica **66** (1980), 237–245.

[14] H. Schaefer, *Banach Lattices and Positive Operators,* Grundlehren Math. Wiss. 215, Springer, 1974.

[15] S.-Y. Shaw, *Mean ergodic theorems and linear functional equations,* Journal of Functional Analysis **87** (1989), 428–441.

[16] S.-Y. Shaw, *Solvability of linear functional equations in Lebesgue spaces,* Publications of the Research Institute for Mathematical Sciences, Kyoto Univ **26** (1990), 691–699.

[17] S.-Y. Shaw, *Non-optimal rates of ergodic limits and approximate solutions,* Journal of Approximation Theory **94** (1998), 285–299.

[18] D. V. Widder, *An Introduction to Transform Theory,* Academic Press, New York and London, 1971.

[19] K. Yosida, *Functional Analysis, 6th ed.,* Springer-Verlag, Berlin, 1980.

[20] A. Zygmund, *Trigonometric Series. Vol. I,* Cambridge University Press, Cambridge, 1959.

[21] Y.-C. Li, R. Sato, and S.-Y. Shaw, *Ratio Tauberian theorems for positive functions and sequences in Banach lattices,* Positivity, to appear.